18.100B Problem Set 3 Solutions
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1) We begin by defining $d : V \times V \rightarrow \mathbb{R}$ such that $d(x, y) = \|x - y\|$. Now to show that this function satisfies the definition of a metric. $d(x, y) = \|x - y\| \geq 0$ and

$$d(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = 0 \iff x = y$$

So the function is positive definite.

$$d(x, y) = \|x - y\| = \| - 1(y - x)\| = -1\|y - x\| = \|y - x\| = d(y, x)$$

Thus the function is symmetric. Finally,

$$d(x, z) = \|x - z\| = \|x - y + y - z\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$$

So the triangle inequality holds. Therefore $d$ is a metric.

2) Once again we must verify the properties of a metric. We have defined $d_1$ as

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

Since $d$ is a metric, it only takes nonnegative values, so $d_1$ cannot be negative. $d_1(x, y)$ is zero exactly when $d(x, y)$ is, so only for $x = y$. Therefore $d_1$ is positive definite. Since $d$ is symmetric, $d_1$ obviously inherits this property. Finally, for $x, y, z \in M$

$$d_1(x, y) + d_1(y, z) = \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} = \frac{d(x, y) + d(y, z) + 2d(x, y)d(y, z)}{1 + d(x, y) + d(y, z) + d(x, y)d(y, z)}$$

$$\geq \frac{1}{1 + d(x, y) + d(y, z) + d(x, y)d(y, z)} \geq 1 - \frac{1}{1 + d(x, z) + d(x, y)d(y, z)} = \frac{d(x, z)}{1 + d(x, z)} = d_1(x, z)$$

So the triangle inequality holds, thus we have a metric. It is easy to see that this metric never takes on a value larger than 1, since $d(x, y) < 1 + d(x, y)$, so under the metric $d_1$, $M$ is bounded.

3) a) $A, B \subseteq M, M$ a metric space. Suppose $x \in A^o \cup B^o$. Without loss of generality, say $x \in A^o$.

Therefore $x$ is an interior point of $A$, so $\exists \varepsilon_1 > 0$ such that the ball of radius $\varepsilon$ centered at $x$ is contained in $A$, or $B_{\varepsilon}(x) \subseteq A$. Since $A \subseteq A \cup B$,

$$B_{\varepsilon}(x) \subseteq A \cup B \implies x \in (A \cup B)^o$$

This shows that $A^o \cup B^o \subseteq (A \cup B)^o$.

b) Now let $x \in A^o \cap B^o$. Therefore $x \in A^o$, so $x$ is an interior point of $A$, hence $\exists \varepsilon_1 > 0$ such that $B_{\varepsilon_1}(x) \subseteq A$. Similarly, $x \in B^o \implies \exists \varepsilon_2 > 0$ such that $B_{\varepsilon_2}(x) \subseteq B$. Let $\delta = \min(\varepsilon_1, \varepsilon_2)$. By the triangle inequality,

$$\delta \leq \varepsilon_1 \implies B_{\delta}(x) \subseteq B_{\varepsilon_1}(x) \Rightarrow B_{\delta}(x) \subseteq A$$

Therefore $B_{\delta}(x) \subseteq A \cap B$, so $x$ is an interior point of $A \cap B$. Hence $A^o \cap B^o \subseteq (A \cap B)^o$. 


Let \( x \in (A \cap B)^\circ \). So \( \exists \varepsilon > 0 \) with \( B_\varepsilon(x) \subseteq A \cap B \). Therefore \( B_\varepsilon(x) \subseteq A \) so \( x \in A^\circ \), and similarly \( x \in B^\circ \). So \( x \in A^\circ \cap B^\circ \). Thus \((A \cap B)^\circ \subseteq A^\circ \cap B^\circ \). So these two sets are equal.

Let \( A = (-1, 0) \) and \( B = [0, 1) \). Then 0 is an interior point of neither \( A \) nor \( B \), so \( 0 \notin A^\circ \cup B^\circ \). But \( A \cup B = (-1, 1) \), so \( 0 \in (A \cup B)^\circ \). Therefore in this instance the two sets are unequal.

4) a) If \( x \in \partial A \) then every ball around \( x \) intersects \( A \) and \( A^c \). Thus \( x \in A \) and \( x \) is a limit point of \( A^c \) or \( x \in A^c \) and \( x \) is a limit point of \( A \). Either way, \( x \in A \cap A^\circ \), and hence \( \partial A \subseteq A \cap A^\circ \).

Now let \( x \in \overline{A} \cap \overline{A^c} \). Since \( x \in \overline{A} \), either \( x \in A \) or \( x \) is a limit point of \( A \), and in both cases any open ball around \( x \) intersects \( A \). Similarly, \( x \in \overline{A^c} \) implies any open ball around \( x \) intersects \( A^c \). Therefore \( x \in \partial A \), so \( \overline{A} \cap \overline{A^c} \subseteq \partial A \). So these two sets are equal.

b) Let \( p \in \partial A \). By a), \( p \in \overline{A} \). Suppose \( p \in A^\circ \) then \( \exists \varepsilon > 0 \) such that \( B_\varepsilon(p) \subseteq A \). But this is an open ball centered at \( p \) which does not intersect \( A^c \), so \( p \notin \partial A \). This contradiction implies that \( p \notin A^\circ \).

Now suppose \( p \in \overline{A} \setminus A^\circ \). For any \( \varepsilon > 0 \), \( p \in \overline{A} \) gives that \( B_\varepsilon(x) \) intersects \( A \), and \( p \notin A^\circ \) implies that \( B_\varepsilon(x) \notin A \), so \( B_\varepsilon(x) \) intersects \( A^c \). So \( p \in \partial A \), and this shows that \( \partial A = \overline{A} \setminus A^\circ \).

c) By a), \( \partial A \) can be written as the intersection of two closed sets. Thus \( \partial A \) is closed.

d) Suppose \( A \) is closed. Then \( \overline{A} = A \), so by a)

\[
\partial A = \overline{A} \cap \overline{A^c} = A \cap \overline{A^c} \subseteq A
\]

Conversely, note that for any set \( B \), if \( x \notin B \) and \( x \notin \partial B \), then there is a positive \( r > 0 \) such that \( B_r(x) \subseteq B^c \) and hence \( x \notin \overline{B} \). This implies that

\[
\text{for any set } B, \overline{B} \subseteq B \cup \partial B.
\]

So if \( \partial A \subseteq A \), then \( \overline{A} \subseteq A \cup \partial A = A \subseteq \overline{A} \) i.e., \( A = \overline{A} \) hence \( A \) is closed.

5) We will show that \( S_r(x) := \{y : d(x, y) = r\} \) is the boundary of \( B_r(x) \). It will follow from the previous exercise that

\[
\overline{B_r(x)} = \partial B_r(x) \cup B_r(x) = \{y : d(x, y) \leq r\}.
\]

It is clear that if \( y \) is such that \( d(x, y) = r \) then \( y \in \partial B_r(x) \) since any ball around \( y \) will have points that are closer to \( x \) and points that are further away. We just have to show that if \( d(x, y) \neq r \), then \( y \) is not in \( \partial B_r(x) \).

But if \( d(x, y) < r \) then for any \( 0 < \varepsilon < r - d(x, y) \) the ball of radius \( \varepsilon \) around \( y \) is all inside \( B_r(x) \) and \( y \notin \partial B_r(x) \); and if \( d(x, y) > r \) then for any \( 0 < \delta < d(x, y) - r \) the ball of radius \( \delta \) around \( y \) is all outside of \( B_r(x) \) so that again \( y \notin \partial B_r(x) \). Thus \( \partial B_r(x) \) is precisely \( S_r(x) \) and we are done.

Here is an example of a different metric space where this result is not true: Consider \( \mathbb{R}^n \) with the discrete metric,

\[
\tilde{d}(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}
\]
and the ball around any point \( p \) with radius 1:

\[
B_1(p) = \{ q : \tilde{d}(p, q) < 1 \} = \{ p \}, \quad \text{while} \quad \{ q : \tilde{d}(p, q) \leq 1 \} = \mathbb{R}^n.
\]

Notice that the open ball is finite and hence closed. In particular, the closure of \( B_1(p) \) is just \{\( p \)\} and not \{\( q : \tilde{d}(p, q) \leq 1 \)\}.

6) We need to show that \( K \) is compact or that every open cover of \( K \) contains a finite subcover. Let \( \{ U_\alpha \}_{\alpha \in A} \) be an open cover of \( K \), so

\[
K = \{ 0, 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots \} \subseteq \bigcup_{\alpha \in A} U_\alpha \implies \exists \alpha_0 \in A \text{ such that } 0 \in U_{\alpha_0}
\]

Since \( U_{\alpha_0} \) is open, \( \exists \varepsilon > 0 \) with \( B_\varepsilon(0) \subseteq U_{\alpha_0} \). Because \( \varepsilon > 0 \), there exists an \( N \in \mathbb{N} \) such that \( n > N \implies \frac{1}{n} < \varepsilon \). Hence the open set \( U_{\alpha_0} \) contains all of \( \{ \frac{1}{n} \} \) with \( n > N \), i.e., it contains all but finitely many elements of \( K \).

Now, for \( i = 1, 2, \ldots, N, \frac{1}{i} \in K \). So \( \exists \alpha_i \in A \) such that \( \frac{1}{i} \in U_{\alpha_i} \). So we have shown that

\[
K \subseteq \bigcup_{i=0}^{N} U_{\alpha_i},
\]

a finite subcover of \( \{ U_\alpha \}_{\alpha \in A} \). So every open cover of \( K \) contains a finite subcover, which shows that \( K \) is compact.

7) We have \( \{ U_\alpha \}_{\alpha \in A} \) an open cover of \( K \). Define

\[
V_{\alpha, n} = \{ x \in U_\alpha | B_{\frac{1}{n}}(x) \subseteq U_\alpha \}^0 \quad \text{for all } \alpha \in A, n \in \mathbb{N}.
\]

The \( U_\alpha \) are open, so for any point \( x \in U_\alpha \), there is some \( n \in \mathbb{N} \) such that

\[
B_{\frac{1}{n}}(x) \subseteq U_\alpha \implies B_{\frac{1}{n}}(x) \subseteq \{ y \in U_\alpha | B_{\frac{1}{n}}(y) \subseteq U_\alpha \} = x \in V_{\alpha, n}. \quad \text{Hence } \bigcup_{n \in \mathbb{N}} V_{\alpha, n} = U_\alpha.
\]

So taking the union over all \( \alpha \in A \), we have

\[
\bigcup_{\alpha \in A} \bigcup_{n \in \mathbb{N}} V_{\alpha, n} = \bigcup_{\alpha \in A} U_\alpha \supseteq K.
\]

So \( \{ V_{\alpha, n} \}_{n \in \mathbb{N}} \) is an open cover of \( K \) (each set is an interior, thus open). By the compactness of \( K \), there exists a finite subcover \( \{ V_{\alpha_i, n_i} \}_{i=1}^{N} \). Let \( \delta = \left( \max_{1 \leq i \leq N} n_i \right)^{-1} \). Then \( \forall x \in K, \exists \nu' \in \{ 1, 2, \ldots, n \} \) with

\[
x \in V_{\alpha_{i'}, n_{i'}} \implies B_{\frac{1}{n_{i'}}}(x) \subseteq U_{\alpha_{i'}}.
\]

Since \( \delta^{-1} = \max_{1 \leq i \leq N} n_i \geq n_{i'} \), we have \( \delta \leq \frac{1}{n_{i'}} \), so \( B_\delta(x) \subseteq B_{\frac{1}{n_{i'}}} \subseteq U_{\alpha_{i'}} \). Thus our \( \delta \) has the prescribed property.