18.100B Problem Set 5
Due Friday October 20, 2006 by 3 PM

Problems:

1) Let \( \mathcal{M} \) be a complete metric space, and let \( X \subseteq \mathcal{M} \). Show that \( X \) is complete if and only if \( X \) is closed.

2) a) Show that a sequence in an arbitrary metric space \( (x_n) \) converges if and only if the ‘even’ and ‘odd’ subsequences \( (x_{2n}) \) and \( (x_{2n-1}) \) both converge to the same limit.

   b) Show that a sequence in an arbitrary metric space \( (x_n) \) converges if and only if the subsequences \( (x_{2n}), (x_{2n-1}), \) and \( (x_{5n}) \) all converge.

3) If \( (x_n) \) and \( (y_n) \) are two bounded sequences of real numbers, show that

   a) \( \lim \sup (x_n + y_n) \leq \lim \sup x_n + \lim \sup y_n \)

   b) \( \lim \inf (x_n + y_n) \geq \lim \inf (x_n) + \lim \inf (y_n) \)

   Moreover, show that if \( (x_n) \) converges, then both inequalities are actually equalities.

   (Hint: Pick a subsequence of \( (x_n + y_n) \) that converges, then, from these \( x_{n_k} \)'s pick a subsequence that converges and do the same for the \( y_{n_k} \)'s)

4) The ‘sequence of averages’ of a sequence of real numbers \( (x_n) \) is the sequence \( (a_n) \) defined by

   \[ a_n = \frac{x_1 + x_2 + \ldots + x_n}{n} \]

   If \( (x_n) \) is a bounded sequence of real numbers, then show that

   \[ \lim \inf x_n \leq \lim \inf a_n \leq \lim \sup a_n \leq \lim \sup x_n. \]

   In particular, if \( x_n \to x \) then show that \( a_n \to x \). Does the convergence of \( (a_n) \) imply the convergence of \( (x_n) \)?

   (Hint: Fix \( \varepsilon > 0 \), let \( x^* = \lim \sup x_n \) and set \( K = \{ k \in \mathbb{N} : x_k \geq x^* + \varepsilon \} \). \( K \) is finite (why?), define \( S_n = \{ i \in \mathbb{N} : i \leq n \} \) and \( T_n = \{ i \in \mathbb{N} : i \notin K \) and \( i \leq n \} \) and define the sequences \( (s_n), (t_n) \) by

   \[ s_n = \sum_{i \in S_n} x_i, \quad t_n = \sum_{i \in T_n} x_i \]

   Explain why \( a_n = \frac{s_n}{n} + \frac{t_n}{n} \to 0 \) and \( \frac{t_n}{n} \leq x^* + \varepsilon \) for any \( n \). Then use the previous exercise to show that \( \lim \sup a_n \leq x^* + \varepsilon \). Hence \( \lim \sup a_n \leq x^* \) (why?)

5) Consider any sequence \( (x_n) \) defined by choosing \( 0 < x_1 < 1 \) and then defining \( x_{n+1} = 1 - \sqrt{1 - x_n} \) for \( n \geq 0 \). Show that \( x_n \) is a decreasing sequence converging to zero. Also, show that \( x_{n+1} - x_n \to \frac{1}{2} \).

6) The Greeks thought that the number \( \Phi \), known as the Golden Mean, was the ratio of the sides of the most aesthetically pleasing rectangles.

   Imagine a line segment \( A \) divided into two smaller line segments \( B \) and \( C \), with lengths \( a, b \), and \( c \) respectively and \( b > c \). If the proportion between \( a \) and \( b \) is the same as the proportion between \( b \) and \( c \), then we call this proportion \( \Phi \).
a) Show that with \( a, b, \) and \( c \) as above, \( \Phi = \frac{b}{c} \) satisfies \( \Phi^2 = \Phi + 1 \). Conclude that \( \Phi = \frac{1+\sqrt{5}}{2} \).

b) Show that:

\[
\Phi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}}
\]

*Hint:* Define \( x_1 = 1 \) and \( x_{n+1} = 1 + \frac{1}{x_n} \).

c) Show that:

\[
\Phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}
\]

*Hint:* Define \( y_1 = 1 \) and \( y_{n+1} = \sqrt{1 + y_n} \).

d) The Fibonacci sequence is defined by \( z_1 = 1, z_2 = 1, \) and \( z_{n+2} = z_{n+1} + z_n \). Show that the sequence of ratios of successive elements, \( \frac{z_{n+1}}{z_n} \), converges to \( \Phi \).

\( \Phi \) shows up a lot in nature. One reason for this might be that it is the ‘most irrational number’.

For more information about this, check out the links section of the course webpage.

**Extra problems:**

1) Prove that \( \lim x_n = x \) if and only if every subsequence of \( (x_n) \) has a subsequence that converges to \( x \).

2) If \( (x_n) \) is a sequence of strictly positive real numbers, show that

\[
\lim \inf \frac{x_{n+1}}{x_n} \leq \lim \inf \sqrt[n]{x_n} \leq \lim \sup \sqrt[n]{x_n} \leq \lim \sup \frac{x_{n+1}}{x_n}
\]

3) Fix a positive number \( \alpha \). Choose \( x_1 > \sqrt{\alpha} \) and define \( x_n \) for \( n > 1 \) by

\[
x_{n+1} = \frac{1}{2} \left( x_n + \frac{\alpha}{x_n} \right)
\]

Prove that \( (x_n) \) decreases monotonically and that \( \lim x_n = \sqrt{\alpha} \). Show that, if \( \varepsilon_n = x_n - \sqrt{\alpha} \), then

\[
\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}
\]

so that, setting \( \beta = 2\sqrt{\alpha} \),

\[
\varepsilon_{n+1} < \beta \left( \frac{\varepsilon_1}{\beta} \right)^{2^n}
\]