Problems:

1) Let $f_n(x) = 1/(nx+1)$ and $g_n(x) = x/(nx+1)$ for $x \in (0, 1)$ and $n \in \mathbb{N}$. Prove that $f_n$ converges pointwise but not uniformly on $(0, 1)$, and that $g_n$ converges uniformly on $(0, 1)$.

2) Let $f_n(x) = x/(1 + nx^2)$ if $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Find the limit function $f$ of the sequence $(f_n)$ and the limit function $g$ of the sequence $(f'_n)$. Prove that $f'(x)$ exists for every $x$ but that $f'(0) \neq g(0)$. For what values of $x$ is $f'(x) = g(x)$? In what subintervals of $\mathbb{R}$ does $f_n \to f$ uniformly? In what subintervals of $\mathbb{R}$ does $f'_n \to g$ uniformly?

3) Let $\mathcal{M}$ be a metric space and $(f_n)$ a sequence of functions defined on a subset $E \subseteq \mathcal{M}$.

   We say that $(f_n)$ is uniformly bounded if there exists a constant $M$ such that $|f_n(x)| \leq M$ for every $n \in \mathbb{N}$ and $x \in E$.

   Prove that if $(f_n)$ is a sequence of bounded real valued functions that converges uniformly to a function $f$, then $(f_n)$ is uniformly bounded. Prove that in this case $f$ is also bounded. If $(f_n)$ is a sequence of bounded functions converging pointwise to $f$, need $f$ be bounded?

4) Prove that if $f_n \to f$ uniformly and $g_n \to g$ uniformly on a set $E$ then
   a) $f_n + g_n \to f + g$ uniformly on $E$.
   b) If each $f_n$ and each $g_n$ is bounded on $E$, prove that $f_n g_n \to fg$ uniformly.

5) Define two sequences $(f_n)$ and $(g_n)$ as follows:

   $$f_n(x) = x \left(1 + \frac{1}{n}\right) \quad \text{if} \quad x \in \mathbb{R}, \ n \geq 1$$

   $$g_n(x) = \begin{cases} 
   \frac{1}{n} & \text{if } x = 0 \text{ or } x \text{ is irrational} \\
   q + \frac{1}{n} & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ in reduced form}
   \end{cases}$$

   Show that, on any interval $[a, b]$ both $f_n$ and $g_n$ converge uniformly, but $f_n g_n$ does not converge uniformly (cf. problem 4b).

6) Assume that $(f_n)$ is a uniformly bounded sequence of functions converging uniformly to $f$ on a set $E$, define $M$ as in problem 3. Let $g$ be continuous on $[-M, M]$, prove that $g \circ f_n \to g \circ f$ uniformly on $E$.

7) a) Show that the sequence of polynomials defined inductively by

   $$P_0(x) = 0$$

   $$P_{n+1}(x) = P_n(x) + \frac{1}{2} (x - P_n^2(x))$$

   converges uniformly on the interval $[0, 1]$ to the function $f(x) = \sqrt{x}$.

   b) Deduce that there exists a sequence of polynomials converging uniformly on $[-1, 1]$ to the function $f(x) = |x|$. 

 Due Friday December 1, 2006 by 3 PM
Extra problems:
Some everywhere continuous, nowhere differentiable functions.
1) (John McCarthy) Consider the function \( g : \mathbb{R} \to \mathbb{R} \) satisfying \( g(x) = g(x + 4) \) for every \( x \), and
\[
g(x) = \begin{cases} 
1 + x & \text{for } -2 \leq x \leq 0 \\
1 - x & \text{for } 0 \leq x \leq 2
\end{cases}
\]
and define
\[
f(x) = \sum_{n=1}^{\infty} 2^{-n} g(2^n x)
\]
Show that \( f \) is continuous. Show that \( f \) is nowhere differentiable as follows: Take \( \Delta x = \pm 2^{-2^k} \), choosing whichever sign makes \( x \) and \( x + \Delta x \) be on the same linear segment of \( g(2^k x) \). Show that
a) \( \Delta (2^n x) = 0 \) for \( n > k \), since \( g(2^n x) \) has period \( 4 \cdot 2^{-2^n} \)
b) \( |\Delta g(2^k x)| = 1 \)
c) \( |\Delta \sum_{n=1}^{k-1} 2^{-n} g(2^n x)| \leq (k - 1) \max |\Delta g(2^n x)| \leq (k - 1) 2^{2k-1} 2^{-2^k} < 2^{k+2^{-k-1}} \)
Conclude that \( |\Delta f/\Delta x| \geq 2^{-k} 2^{2k} - 2^{k+2^{-k-1}} \) which goes to infinity with \( k \), and hence \( f \) is nowhere differentiable.

2) (Van der Waerden following Billingsley) Let \( a_0(x) \) denote the distance from \( x \) to the nearest integer, \( a_k(x) = 2^{-k} a_0(2^k x) \), and define
\[
f(x) = \sum a_k(x).
\]
a) Prove that \( f \) is everywhere continuous.
b) Prove that if a function \( h \) has a derivative at \( x \) and \( u_n \leq x \leq v_n \) are such that \( u_n < v_n \) and \( u_n - v_n \to 0 \) then
\[
\frac{h(v_n) - h(u_n)}{v_n - u_n} \to h'(x)
\]
c) Prove that \( f \) is nowhere differentiable as follows: Notice that if \( u \) is a dyadic number of order \( n \) (i.e., of the form \( \frac{i}{2^n} \) for some integer \( i \) ) then \( 2^k u \) is an integer for \( k \geq n \) and
\[
f(u) = \sum_{k=0}^{n-1} a_k(u).
\]
Fix \( x \) and let \( u_n, v_n \) be successive dyadics of order \( n \) (i.e., \( v_n - u_n = 2^{-n} \)) such that \( u_n \leq x < v_n \). Show that
\[
\frac{f(v_n) - f(u_n)}{v_n - u_n} = \sum_{k=0}^{n-1} \frac{a_k(v_n) - a_k(u_n)}{v_n - u_n}
\]
Show that each term on the right hand side is either a 1 or a \(-1\) and conclude that the left hand side can not converge to a finite limit.

For more examples, see the Related Resources section of the course.