1

Let $s_n = \sum_{i=1}^{n} x_i$ and $\sigma_n = \sum_{i=1}^{n} y_i$ be the partial sums. Then we claim that $\sigma_n$ is the average of the first $n$ $s_n$, i.e.

$$\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}$$

To see this, we will look at the "coefficients" of the $x_k$ in the expressions for the $y_j$; if this makes you nervous, think of them as variables, for which we will later plug in values to get actual sequences of real numbers. If $k > j$, then $x_k$ does not appear in the expression for $y_j$, while if $k \leq j$ then $x_k$ appears with a coefficient of $(k - 1)/(j(j - 1))$, where this expression should be interpreted as equal to 1 in the degenerate case $k = j = 1$. Note that $1/(j(j - 1)) = 1/(j - 1) - 1/j$. Using this, we can determine the coefficient of $x_k$ in $\sigma_n$; this is zero if $j > n$, and if $j \leq n$, it is equal to

$$\sum_{j=k}^{n} \frac{k-1}{j(j-1)} = (k-1)\sum_{j=k}^{n} \left( \frac{1}{j-1} - \frac{1}{j} \right) = (k-1)(\frac{1}{k-1} - \frac{1}{n}) = \frac{n - k + 1}{n}$$

Where we got rid of the sum by noting that the negative part of one term of the sum is equal to the positive part of the next term, and hence the sum "telescopes" to $1/(k-1) - 1/n$. But now consider the sum of partial sums $s_1 + s_2 + \cdots + s_n; x_k$ will appear precisely $n - k + 1$ times as long as $k \leq n$, and hence the coefficient of $x_k$ in $(s_1 + s_2 + \cdots + s_n)/n$ is $(n - k + 1)/n$, which is exactly the coefficient we computed above for $\sigma_n$, and hence $\sigma_n$ is the average of the $s_n$ as desired.
Fact: Suppose \( \{s_n\} \) is a sequence of real numbers with \( s_n \to s \) as \( n \to \infty \). Then if \( \sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n} \), then \( \sigma_n \to \infty \) as \( n \to \infty \).

Proof: The idea is to split the sum determining \( \sigma_n \) into two parts; one part will be small because the denominator is large, and the other part will be close to \( s \). Fix \( \epsilon > 0 \), let \( N \in \mathbb{N} \) be sufficiently large that for \( n > N \) we have \( |s - s_n| < \epsilon \). With this \( N \) fixed, choose \( M >> N \) such that \( (|\sum_{i=1}^{N} s_i|)/M < \epsilon \) and such that \( N|s|/M < \epsilon \). Note that both these inequalities will continue to hold for \( n > M \). For any such \( n \), we compute

\[
|\sigma_n - s| = \frac{|\sum_{i=1}^{N} s_i + \sum_{i=N+1}^{n} s_i - ns|}{n} \\
\leq \frac{|\sum_{i=1}^{N} s_i|}{n} + \frac{|\sum_{i=N+1}^{n} s_i - (n-N)s|}{n} + \frac{N|s|}{n} \\
< \epsilon + \frac{|\sum_{i=N+1}^{n} s_i - s|}{n} + \epsilon < 2\epsilon + \frac{(n-N)\epsilon}{n} < 3\epsilon
\]

Thus \( \sigma_n \to s \) as \( n \to \infty \) and the claim is proved.

Now take the alternating series \( 1 - 1 + 1 - 1 + \cdots \). This has partial sums \( s_i = 1 \) if \( i \) is odd, and \( 0 \) if \( i \) is even. Thus taking averages, we have \( \sigma_n = 1/2 \) if \( n \) is even, and \( (n+1)/(2n) \) if \( n \) is odd. Since \( \lim_{n \to \infty} (n+1)/(2n) = 1/2 \), we see that \( \sigma_n \to 1/2 \).

2

We define \( f(x) = 1/2(x + \alpha/x) \) for \( x \in \mathbb{R} \). We have

Claim: suppose \( x > \sqrt{\alpha} \). Then \( x > f(x) > \sqrt{\alpha} \).

Proof: We have

\[
x - f(x) = x - 1/2(x - \alpha/x) = 1/2(x - \alpha/x) = 1/2((x^2 - \alpha)/x) > 0
\]
Since $x^2 > \alpha > 0$. For the other inequality, we will show that $f(x)^2 > \alpha$. We compute

$$\left(\frac{1}{2}x + \frac{\alpha}{x}\right)^2 = \frac{\alpha}{2} + \frac{1}{4}(x^2 + \frac{\alpha^2}{x^2}) > \frac{\alpha}{2} + \frac{1}{2}\sqrt{\frac{x^2\alpha^2}{x^2}} = \alpha$$

We used the AM-GM inequality, which states that for any positive real numbers $a$ and $b$, $(a + b)/2 > \sqrt{ab}$, with equality if and only if $a = b$; this can be proved by noting that $(\sqrt{a} - \sqrt{b})^2 \geq 0$, with equality if and only if $a = b$. This proves the claim.

Now, we start with some $x_1 > \alpha$, and we inductively define $x_{n+1} = f(x)$. Then by the claim, $(x_n)$ is a decreasing sequence, bounded from below by $\sqrt{\alpha}$. By Rudin Theorem 3.14, this sequence converges to some $x$. Since $x$ is the inf of $(x_i)$, we must have $x \geq \sqrt{\alpha} > 0$. Hence by Rudin Theorem 3.3, we have $\lim_{n \to \infty} 1/x_n = 1/x$. Applying this Theorem repeatedly, we then have $\alpha/x_n \to \alpha/x$, then $x_n + \alpha/x_n \to x + \alpha/x$, and finally $1/2(x_n + \alpha/x_n) \to 1/2(x + \alpha/x)$. In other words, we have $f(x_n) \to f(x)$ as $x \to \infty$ (readers who are familiar with the concept will note that this argument amounts to proving the continuity of $f$). But by the definition of $(x_n)$, we have

$$f(x) = \lim_{n \to \infty} f(x) = \lim_{n \to \infty} x_{n+1} = x$$

In other words, $x = 1/2(x + \alpha/x)$, or $x^2 = \alpha$. This means that $x = \pm \sqrt{\alpha}$, but since $x \geq \sqrt{\alpha}$, we have $x = \sqrt{\alpha}$.

3

We have an alternating sequence $(c_i)$, i.e. $c_{2k} > 0$ and $c_{2k+1} < 0$ for all $k \in \mathbb{N}$, with the property that $|c_k| > |c_{k+1}|$ and $|c_k| \to 0$ as $k \to \infty$. We have the partial sums $s_n = \sum_{k=1}^{n} c_k$, and we wish to show that $(s_n)$ converges. Suppose $n, m \in \mathbb{N}$, and that $n > m$. When comparing $s_n$ and $s_m$, there are four cases, depending on the parity of $m$ and $n$.

Case 1: $n$ and $m$ are odd. Then $s_n > s_m$. Indeed, note that $s_{m+2} - s_m = c_{m+1} + c_{m+2}$. Since $m + 1$ is even, $c_{m+1} > 0$, and $c_{m+1} > |c_{m+2}|$, so $c_{m+1} + c_{m+2} > 0$. 

3
Case 2: $n$ and $m$ are even. Then $s_n < s_m$. The argument is the same as Case 1.

Case 3: $n$ is odd and $m$ is even. Then $s_m > s_n$. Indeed, by Case 2 $s_m > s_{n+1}$. But $s_{n+1} - s_n = c_{n+1} > 0$.

Case 4: $n$ is even and $m$ is odd. Then $s_n > s_m$. Same argument as Case 3.

Putting these together, we see that $s_1 < s_3 < \cdots$ is an increasing sequence, bounded above, $s_2 > s_4 > \cdots$ is a decreasing sequence, bounded below, and that if $k$ is even and $j$ is odd then $s_k > s_j$. Thus by Rudin Theorem 3.14 there are real numbers $r, s$ with $\lim_{k \to \infty} s_{2k} = r$ and $\lim_{k \to \infty} s_{2k+1} = s$.

If $r = s$, then we are done, since for any $\varepsilon > 0$, take $N$ sufficiently large such that for any $k > N$, then both $|s - s_{2k}| < \varepsilon$ and $|s_{2k+1}| < \varepsilon$; then $2N$ will work for this $\varepsilon$ and $s_n \to s$.

So suppose $s > r$. Pick $\delta < (r - s)/2$ and $k$ odd and sufficiently large that $\delta > c_{k+1} > 0$. Then $s_k \leq r$ and $s_{k+1} \geq s$ and so

$$c_{k+1} = s_{k+1} - s_k \geq s - r > \delta > c_{k+1}$$

Which is a contradiction.

4

We are looking for an explicit rearrangement of the convergent series $1 - 1/2 + 1/3 - 1/4 + \cdots$ that does not converge. We will construct a rearrangement whose partial sums go to infinity as $n \to \infty$.

Let $x_k = 1/(2k + 1)$ and $s_n = \sum_{k=0}^{n} x_n$. Then $(s_n)$ is an increasing se-
Now we pick an increasing sequence $N_1, N_2, \ldots$ of positive integers as follows. Set $N_1 = 1$. Having chosen $N_1, \ldots, N_k$, we pick an $N_{k+1}$ such that $s_{N_{k+1}} - s_{N_k} > 1/2$; since $s_n \to \infty$, this is always possible.

In fact, we can be a little more precise in our choice of $N_k$. An slight refinement of the above argument shows that, for $n > m$, we have $2(s_n - s_m) > s'_{2n+1} - s_{2m}$. But the proof of the divergence of the harmonic series shows that $s'_{2k+1} - s'_{2k} > 1/2$; indeed, this sum has $2^k$ terms in it, each of which is larger than $1/2^{k+1}$. Thus if we take $N_k = 2^{2(k-1)}$, we will have

$$s_{N_{k+1}} - s_{N_k} > \frac{1}{2}(s_{2^{2k+1}} - s_{2^{2k-1}}) > \frac{1}{2}$$

as desired.

We can now describe our divergent rearrangement of the sum:

$$1 - \frac{1}{2} + \sum_{j=N_1+1}^{N_2} \frac{1}{2j+1} - \frac{1}{4} + \cdots + \sum_{j=N_k+1}^{N_{k+1}} \frac{1}{2j+1} - \frac{1}{2k} + \cdots$$

By construction, the negative terms of this sum are very sparse, occurring only at numbers of the form $N_k + k$; for notational convenience, we define $M_k = N_k + k$. Let $t_n$ be the $n$'th partial sum; our goal is to prove $\lim_{n \to \infty} t_n = \infty$.

Fact 1: if $M_k < n < M_{k+1}$, then $t_{M_k} < t_n$
Proof: $t_n - t_{M_k} = \sum_{j=N_k+1}^{n-k} 1/(2j + 1) > 0$

Fact 2: for $k \geq 1$, $t_{M_{k+1}} - t_{M_k} > 1/4$

Proof:

$$t_{M_{k+1}} - t_{M_k} = \sum_{j=N_{k+1}}^{N_{k+1}} 1/(2j + 1) - 1/(2k + 2) > 1/2 - 1/(2k + 2) > 1/4$$

Fact 3: $t_{M_k} > k/2$

Proof: $t_{M_1} = 1/2$, so the result follows from Fact 2 and induction.

With these facts, we can show that $t_n$ diverges. Let $r$ be any real number, and pick $k \in \mathbb{N}$ such that $k/2 > r$. Then I claim that for $n > M_k$, $t_n > k/2 > r$. To see this, for any such $n$ take $k'$ such that $M_{k'} \leq n < M_{k'+1}$; then by fact 1 $t_n \geq t_{M_{k'}}$. We must have $k' \geq k$, since $n > M_k$, and so by fact 3 $t_{M_{k'}} > k'/2 \geq k/2 > r$. This proves that $t_k \to \infty$ as $k \to \infty$. 