Problem Set 8

1. Let \((f_n)\) and \((g_n)\) be two sequences of functions \([a, b] \to \mathbb{R}\), each of which converges uniformly,
\[
\lim_{n \to \infty} f_n = f, \quad \lim_{n \to \infty} g_n = g.
\]
Suppose that \(f\) and \(g\) are bounded. Show that then, \((f_n g_n)\) also converges uniformly to \(fg\). Please write your solution to this problem out clearly in LaTeX (3 points).

2. We consider continuous functions \(f : [0, 1] \to \mathbb{R}\) such that \(f(0) = 0\) and \(f(1) = 1\). Given such a function \(f\), define another function \(\hat{f}\) by
\[
\hat{f}(x) = \begin{cases} 
\frac{1}{4} f(2x) & \text{if } x < 1/2, \\
\frac{3}{4} f(2x - 1) + \frac{1}{4} & \text{if } x \geq 1/2.
\end{cases}
\]
Prove that \(\hat{f}\) belongs to the same class of functions. Next, prove that \(d(\hat{f}, \hat{g}) \leq \frac{3}{4} d(f, g)\), where \(d(f, g) = \max\{\|f(x) - g(x)\| : 0 \leq x \leq 1\}\). Then, prove that there is exactly one continuous function \(f\) in our class such that \(\hat{f} = f\). (It’s fun to try to graph it.) (5 points)

3. Let \((f_n)\) be a sequence of functions \([a, b] \to \mathbb{R}\) such that: (i) \(f_n(x) \leq 0\) if \(n\) is even, \(f_n(x) \geq 0\) if \(n\) is odd; (ii) \(|f_n(x)| \geq |f_{n+1}(x)|\) for all \(x\); (iii) \(f_n\) converges to 0 uniformly. Prove that then,
\[
\sum_{n} f_n
\]
is uniformly convergent. (5 points)

Total: 3+5+5 = 13 points.