Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces, and \(f : X \to Y\) a map.

**Definition 12.1.** \(f\) is continuous (everywhere) if: whenever \((x_n)\) is a sequence in \(X\) converging to some point \(p \in X\), then \((f(x_n))\) converges to \(f(p)\) in \(Y\).

**Definition 12.2.** \(f\) is continuous (everywhere) if: for any open subset \(V \subset Y\), the preimage \(f^{-1}(V) = \{x \in X : f(x) \in V\}\) is an open subset of \(X\).

**Definition 12.3.** \(f\) is continuous (everywhere) if: for all \(p \in X\) and all \(\epsilon > 0\), there is a \(\delta > 0\) such that if \(d_X(x, p) < \delta\) then \(d_Y(f(x), f(p)) < \epsilon\).

The “such that...” part can be reformulated as follows: “\(f(B_\delta(x)) \subset B_\epsilon(f(x))\)”.

Or as follows: “\(B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))\)”.

**Definition 12.4.** \(f\) is continuous (everywhere) if: for any closed subset \(W \subset Y\), the preimage \(f^{-1}(W)\) is a closed subset of \(X\).

**Theorem 12.5.** The four definitions above are equivalent.

**Theorem 12.6.** If \(f : X \to Y\) and \(g : Y \to Z\) are continuous, then the composition \(g \circ f : X \to Z\) is continuous.

**Corollary 12.7.** If \(f, g : X \to \mathbb{R}\) (with the usual metric on the real numbers) are continuous, then \(f(x) + g(x)\) and \(f(x)g(x)\) are continuous.

**Corollary 12.8.** If \(f : X \to \mathbb{R}\) is continuous and everywhere nonzero, then \(1/f\) is continuous.

**Theorem 12.9.** If \(f : X \to Y\) is continuous and \(K \subset X\) is compact, then \(f(K) \subset Y\) is compact.

**Corollary 12.10.** If \(X\) is a compact metric space and \(f : X \to \mathbb{R}\) a continuous function, then \(f\) is bounded and has a minimum and maximum.

**Corollary 12.11.** Let \(X\) be a compact metric space, and \(f : X \to Y\) a map which is continuous, one-to-one, and onto. Then the inverse map \(f^{-1} : Y \to X\) (defined by \(f(x) = y \iff x = f^{-1}(y)\)) is again continuous.

We return to basic definitions. Let \(f : X \to Y\) be a map between metric spaces. Fix a point \(p \in X\).

**Definition 12.12.** \(f\) is continuous at \(p\) if: whenever \((x_n)\) is a sequence in \(X\) converging to (our particular point) \(p\), then \((f(x_n))\) converges to \(f(p)\).
Definition 12.13. $f$ is continuous at $p$ if: for all $\epsilon > 0$, there is a $\delta > 0$ such that if $d_X(x, p) < \delta$ then $d_Y(f(x), f(p)) < \epsilon$.

Theorem 12.14. The two definitions above are equivalent.

Definition 12.15. Let $X, Y$ be metric spaces, $f : X \to Y$ a map, and $p \in X$ a point. We write

$$\lim_{x \to p} f(x) = q \in Y$$

if the following holds: for all $\epsilon > 0$, there is a $\delta > 0$ such that if $x \neq p$ and $d_X(x, p) < \delta$, then $d_Y(f(x), f(p)) < \epsilon$.

The advantage of this is that it makes sense even if $f$ is defined only on $X \setminus \{p\}$.

Lemma 12.16. If $f : X \to Y$ satisfies $\lim_{x \to p} f(x) = f(p)$, then it is continuous at $p$ (the converse also holds).