18.100C Lecture 15 Summary

**Theorem 15.1.** Suppose that \( f \) and \( g \) are functions satisfying \( f(g(x)) = x \). Take a point \( p \) in the interior of the domain of definition of \( g \), and such that \( f(x) \) lies in the interior of the domain of definition of \( f \). Suppose that there is some \( \delta > 0 \) such that \( g \) is increasing on the interval \( (p - \delta, p + \delta) \), and that \( g'(p) \) exists and is positive (alternatively, \( g \) could be strictly decreasing and \( g'(p) \) could be negative). Then \( f \) is differentiable at \( g(p) \), and

\[
f'(g(p)) = \frac{1}{g'(p)}.
\]

Only differentiability needs to be proved; the formula for the derivative then follows from the chain rule.

**Example 15.2.** \( f(x) = \log(x) \) is differentiable for all \( x > 0 \), and \( f'(x) = 1/x \).

**Example 15.3.** For any natural number \( n \), the function \( f(x) = x^{1/n} \) is differentiable for all \( x > 0 \), and \( f'(x) = (1/n)x^{1/n-1} \).

Definition of higher differentiability. The rest of this lecture is about forms of Taylor’s theorem.

**Theorem 15.4.** Suppose that \( f \) is \( m \) times differentiable at \( p \). Then one can write

\[
f(x) = f(p) + (x-p)f'(p) + \frac{(x-p)^2}{2}f''(p) + \cdots + \frac{(x-p)^m}{m!}f^{(m)}(p) + r(x)(x-p)^m,
\]

where \( \lim_{x \to p} r(x) = 0 \).

Equivalently:

**Theorem 15.5.** Suppose that \( f \) is \( m \) times differentiable at \( p \). Then for each \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that if \( |x-p| < \delta \), then

\[
\left| f(x) - f(p) - (x-p)f'(p) - \frac{(x-p)^2}{2}f''(p) - \cdots - \frac{(x-p)^m}{m!}f^{(m)}(p) \right| \leq \epsilon |x-p|^m.
\]

**Theorem 15.6.** Suppose that \( f \) is \( m \) times differentiable in the (closed) interval bounded by \( a \) and \( b \); that \( f^{(m)} \) is continuous in the same interval; and that \( f^{(m+1)} \) exists at all interior points of that interval. Then

\[
f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2}f''(a) + \cdots + \frac{(b-a)^m}{m!}f^{(m)}(a) + \frac{(b-a)^{m+1}}{(m+1)!}f^{(m+1)}(x)
\]

for some point \( x \) in the interior of the interval bounded by \( a \) and \( b \).