Definition of Riemann-Stieltjes (RS) integral (of a bounded function, with respect to a nondecreasing function \( \alpha \)).

**Example 20.1.** Constant functions are always RS integrable, and

\[
\int_a^b c \, d\alpha = c(\alpha(b) - \alpha(a)).
\]

**Example 20.2.** Take some \( x_\ast \in (a, b) \), and define \( \alpha \) to be the jump function

\[
\alpha(x) = \begin{cases} 
0 & x < x_\ast, \\
1 & x \geq x_\ast.
\end{cases}
\]

\( f \) is RS-integrable with respect to \( \alpha \) if \( \lim_{x \to x_\ast} f(x) = f(x_\ast) \) holds (in particular, this is true if \( f \) is continuous at \( x_\ast \)). In that case,

\[
\int_a^b f(x) \, d\alpha = f(x_\ast).
\]

**Theorem 20.3.** (i) \( f \) is RS-integrable if and only if: for every \( \epsilon > 0 \), there is a partition \( P \) such that

\[
S(f, \alpha, P) - s(f, \alpha, P) < \epsilon.
\]

(ii) Suppose that \( P \) is a partition as in (i). For each \( i \), take a point \( x_i^\ast \in [x_{i-1}, x_i] \). Then

\[
\left| \sum_i f(x_i^\ast) \Delta \alpha_i - \int_a^b f \, d\alpha \right| < \epsilon.
\]

**Theorem 20.4.** Continuous functions \( f \) are RS-integrable for any \( \alpha \).

**Theorem 20.5.** If \( (f_n) \) are RS-integrable with respect to \( \alpha \), and \( f_n \to f \) uniformly, then \( f \) is RS-integrable for the same \( \alpha \), and

\[
\int_a^b f \, d\alpha = \lim_{n \to \infty} \int_a^b f_n \, d\alpha.
\]

**Theorem 20.6.** (i) If \( f \) and \( g \) are RS-integrable, then \( f + g \) is RS-integrable, and

\[
\int_a^b f + g \, d\alpha = \int_a^b f \, d\alpha + \int_a^b g \, d\alpha.
\]

(ii) If \( f \) is RS-integrable and \( c \) is a constant, then \( cf \) is RS-integrable, and

\[
f_a^b cf \, d\alpha = cf_a^b f \, d\alpha.
\]

(iii) If \( f \) is RS-integrable and \( f(x) \geq 0 \) for all \( x \), then \( f_a^b f \, d\alpha \geq 0 \).