Theorem 21.1. If $f$ is RS-integrable and $\phi$ is continuous (on some closed interval containing all values $f(x)$, $x \in [a,b]$), then $\phi(f)$ is RS-integrable (for the same $\alpha$).

Corollary 21.2. If $f$ and $g$ are RS-integrable, then $fg$ is RS-integrable (for the same $\alpha$).

Corollary 21.3. If $f$ is RS-integrable, then $|f|$ is RS-integrable, and $\int_a^b |f| \, d\alpha \leq \int_a^b |f| \, d\alpha$.

The following is easy:

Theorem 21.4. Suppose that $\phi$ is strictly increasing and continuous, and maps $[A, B]$ to $[a, b]$. Then if $f : [a, b] \to \mathbb{R}$ is RS-integrable for some $\alpha$, $g = f(\phi) : [A, B] \to \mathbb{R}$ is RS-integrable for $\beta = \alpha(\phi)$, and

$$\int_a^b f \, d\alpha = \int_A^B g \, d\beta.$$

But this is hard:

Theorem 21.5. Suppose that $\alpha$ is everywhere differentiable, and $\alpha'$ is Riemann-integrable. Let $f$ be a function which is R-S integrable for $\alpha$. Then $f(x)\alpha'(x)$ is Riemann-integrable, and

$$\int_a^b f(x)\alpha'(x) \, dx = \int_a^b f \, d\alpha.$$

Together they yield the following form of the substitution rule:

Corollary 21.6. Suppose that $\phi$ is strictly increasing, differentiable, maps $[A, B]$ to $[a, b]$, and that $\phi'$ is Riemann integrable. Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Then $f(\phi(x))\phi'(x) : [A, B] \to \mathbb{R}$ is again Riemann integrable, and

$$\int_A^B f(\phi(x))\phi'(x) \, dx = \int_a^b f(x) \, dx.$$