Convergent sequences in metric spaces. Examples.

**Theorem 8.1.** Let \((x_n)\) be a convergent sequence, where all the \(x_n\) lie in a subset \(E \subset X\). Then the limit \(x\) lies in \(E\).

**Theorem 8.2.** If \(x \in \bar{E}\), there is a sequence \((x_n)\), \(x_n \in E\), which converges to \(x\).

Subsequence of a convergent sequence is convergent (same limit).

**Theorem 8.3.** Let \((X, d)\) be a compact metric space. Then every sequence \((x_n)\) in \(X\) has a convergent subsequence.

**Corollary 8.4.** Every bounded sequence in \(\mathbb{R}^d\) has a convergent subsequence.

Definition of Cauchy sequence. Every convergent sequence is a Cauchy sequence.

**Lemma 8.5.** Let \((x_n)\) be a Cauchy sequence. If it has a convergent subsequence, then \((x_n)\) itself converges (to the same point).

**Theorem 8.6.** Let \((X, d)\) be a compact metric space. Then every Cauchy sequence converges.

**Corollary 8.7.** Every Cauchy sequence in \(\mathbb{R}^n\) converges.

A metric space where this happens (every Cauchy sequence converges) is called complete. So, we just showed that compact metric spaces as well as \(\mathbb{R}^n\) are complete.