Lecture 13

Let $A$ be an open set in $\mathbb{R}^n$, and let $f : A \rightarrow \mathbb{R}$ be a continuous function. For the moment, we assume that $f \geq 0$. Let $D \subseteq A$ be a compact and rectifiable set. Then $f|D$ is bounded, so $\int_D f$ is well-defined. Consider the set of all such integrals:

$$# = \{ \int_D f : D \subseteq A, D \text{ compact and rectifiable} \}. \quad (3.122)$$

**Definition 3.22.** The improper integral of $f$ over $A$ exists if $*$ is bounded, and we define the improper integral of $f$ over $A$ to be its l.u.b.

$$\int_A^# f \equiv \text{l.u.b. } \int_D f = \text{improper integral of } f \text{ over } A. \quad (3.123)$$

**Claim.** If $A$ is rectifiable and $f : A \rightarrow \mathbb{R}$ is bounded, then

$$\int_A^# f = \int_A f. \quad (3.124)$$

**Proof.** Let $D \subseteq A$ be a compact and rectifiable set. So,

$$\int_D f \leq \int_A f \quad (3.125)$$

$$\Rightarrow \sup_D \int_D f \leq \int_A f \quad (3.126)$$

$$\Rightarrow \int_A^# f \leq \int_A f. \quad (3.127)$$

The proof of the inequality in the other direction is a bit more complicated.

Choose a rectangle $Q$ such that $\bar{A} \subseteq \text{Int } Q$. Define $f_A : Q \rightarrow \mathbb{R}$ by

$$f_A(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \quad (3.128)$$

By definition,

$$\int_A f = \int_Q f_A. \quad (3.129)$$

Now, let $P$ be a partition of $Q$, and let $R_1, \ldots, R_k$ be rectangles belonging to a partition of $A$. If $R$ is a rectangle belonging to $P$ not contained in $A$, then $R - A \neq \phi$. In such a case, $m_R(f_A) = 0$. So

$$L(f_A, P) = \sum_{i=1}^k m_{R_i}(f_A)v(R_i). \quad (3.130)$$
On the rectangle $R_i,$

$$f_A = f \geq m_{R_i}(f_A). \quad (3.131)$$

So,

$$\sum_{i=1}^{k} m_{R_i}(f_A)v(R_i) \leq \sum_{i=1}^{k} \int_{R_i} f = \int_{D} f \leq \int_{A}^\#, \quad (3.132)$$

where $D = \bigcup R_i,$ which is compact and rectifiable.

The above was true for all partitions, so

$$\int_{Q} f_A \leq \int_{Z} f. \quad (3.133)$$

We proved the inequality in the other direction, so

$$\int_{A} f = \int_{A}^\#. \quad (3.134)$$

$\square$

### 3.8 Exhaustions

**Definition 3.23.** A sequence of compact sets $C_i, i = 1, 2, 3 \ldots$ is an *exhaustion of A* if $C_i \subseteq \text{Int } C_{i+1}$ for every $i,$ and $\bigcup C_i = A.$

It is easy to see that

$$\bigcup \text{Int } C_i = A. \quad (3.135)$$

Let $C_i, i = 1, 2, 3 \ldots$ be an exhaustion of $A$ by compact rectifiable sets. Let $f : A \to \mathbb{R}$ be continuous and assume that $f \geq 0.$ Note that

$$\int_{C_i} f \leq \int_{C_{i+1}} f, \quad (3.136)$$

since $C_{i+1} \supseteq C_i.$ So

$$\int_{C_i} f, \ i = 1, 2, 3 \ldots \quad (3.137)$$

is an increasing (actually, non-decreasing) sequence. Hence, either $\int_{C_i} f \to \infty$ as $i \to \infty,$ or it has a finite limit (by which we mean $\lim_{i \to \infty} \int_{C_i} f$ exists).
Theorem 3.24. The following two properties are equivalent:

1. $\int_A^# f$ exists,

2. $\lim_{i \to \infty} \int_{C_i} f$ exists.

Moreover, if either (and hence both) property holds, then

$$\int_A^# f = \lim_{i \to \infty} \int_{C_i} f.$$  \hfill (3.138)

Proof. The set $C_i$ is a compact and rectifiable set contained in $A$. So, if

$$\int_A^# f$$ exists, then

$$\int_{C_i} f \leq \int_A^# f.$$  \hfill (3.140)

That shows that the sets

$$\int_{C_i} f, \ i = 1, 2, 3 \ldots$$  \hfill (3.141)

are bounded, and

$$\lim_{i \to \infty} \int_{C_i} f \leq \int_A^# f.$$  \hfill (3.142)

Now, let us prove the inequality in the other direction.

The collection of sets \{Int $C_i$ : $i = 1, 2, 3 \ldots$\} is an open cover of $A$. Let $D \subseteq A$ be a compact rectifiable set contained in $A$. By the H-B Theorem,

$$D \subseteq \bigcup_{i=1}^N \text{Int } C_i,$$  \hfill (3.143)

for some $N$. So, $D \subseteq \text{Int } C_N \subseteq C_N$. For all such $D$,

$$\int_D f \leq \int_{C_i} f \leq \lim_{i \to \infty} \int_{C_i} f.$$  \hfill (3.144)

Taking the infimum over all $D$, we get

$$\int_A^# f \leq \lim_{i \to \infty} \int_{C_i} f.$$  \hfill (3.145)

We have proved the inequality in both directions, so

$$\int_A^# f = \lim_{i \to \infty} \int_{C_i} f.$$  \hfill (3.146)
A typical illustration of this theorem is the following example. Consider the integral
\[ \int_0^1 \frac{dx}{\sqrt{x}}, \] (3.147)
which we wrote in the normal integral notation from elementary calculus. In our notation, we would write this as
\[ \int_{(0,1)} \frac{1}{\sqrt{x}}. \] (3.148)
Let \( C_N = [\frac{1}{N}, 1 - \frac{1}{N}] \). Then
\[ \int_{(0,1)}^{\#} \frac{1}{\sqrt{x}} = \lim_{N \to \infty} \int_{C_N} \frac{A}{\sqrt{x}} \] (3.149)
\[ = 2\sqrt{x}|_{1/1/N}^{1-1/N} \to 2 \text{ as } N \to \infty. \]
So,
\[ \int_{(0,1)}^{\#} \frac{1}{\sqrt{x}} = 2. \] (3.150)
Let us now remove the assumption that \( f \geq 0 \). Let \( f : A \to \mathbb{R} \) be any continuous function on \( A \). As before, we define
\[ f_+(x) = \max\{f(x), 0\}, \] (3.151)
\[ f_-(x) = \max\{-f(x), 0\}. \] (3.152)
We can see that \( f_+ \) and \( f_- \) are continuous.

**Definition 3.25.** The improper R. integral of \( f \) over \( A \) exists if and only if the improper R. integral of \( f_+ \) and \( f_- \) over \( A \) exist. Moreover, we define
\[ \int_A^{\#} f = \int_A^{\#} f_+ - \int_A^{\#} f_. \] (3.153)
We compute the integral using an exhaustion of \( A \).
\[ \int_A^{\#} f = \lim_{N \to \infty} \left( \int_{C_N} f_+ - \int_{C_N} f_- \right) \] (3.154)
\[ = \lim_{N \to \infty} \int_{C_N} f. \]
Note that \( |f| = f_+ + f_- \), so
\[ \lim_{N \to \infty} \left( \int_{C_N} f_+ + \int_{C_N} f_- \right) = \lim_{N \to \infty} \int_{C_N} |f|. \] (3.155)
Therefore, the improper integral of $f$ exists if and only if the improper integral of $|f|$ exists.

Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ e^{-1/x} & \text{if } x > 0. \end{cases} \quad (3.156)$$

This is a $C^\infty(\mathbb{R})$ function. Clearly, $f'(x) = f''(x) = \ldots = 0$ when $x = 0$, so in the Taylor series expansion of $f$ at zero,

$$\sum a_n x^n = 0, \quad (3.157)$$

all of the coefficients $a_n$ are zero. However, $f$ has a non-zero value in every neighborhood of zero.

Take $a \in \mathbb{R}$ and $\epsilon > 0$. Define a new function $g_{a,a+\epsilon} : \mathbb{R} \to \mathbb{R}$ by

$$g_{a,a+\epsilon}(x) = \frac{f(x - a)}{f(x - a) + f(a + \epsilon - x)}. \quad (3.158)$$

The function $g_{a,a+\epsilon}$ is a $C^\infty(\mathbb{R})$ function. Notice that

$$g_{a,a+\epsilon} = \begin{cases} 0 & \text{if } x \leq a, \\ 1 & \text{if } x \geq a + \epsilon. \end{cases} \quad (3.159)$$

Take $b$ such that $a < a + \epsilon < b - \epsilon < b$. Define a new function $h_{a,b} \in C^\infty(\mathbb{R})$ by

$$h_{a,b}(x) = g_{a,a+\epsilon}(x)(1 - g_{a-\epsilon,b}(x)). \quad (3.160)$$

Notice that

$$h_{a,b} = \begin{cases} 0 & \text{if } x \leq a, \\ 1 & \text{if } a + \epsilon \leq x \leq b - \epsilon, \\ 0 & \text{if } b \leq x. \end{cases} \quad (3.161)$$