Lecture 15

We restate the partition of unity theorem from last time. Let \( \{U_\alpha : \alpha \in I\} \) be a collection of open subsets of \( \mathbb{R}^n \) such that
\[
U = \bigcup_{\alpha \in I} U_\alpha. \tag{3.169}
\]

**Theorem 3.30.** There exist functions \( f_i \subseteq C^\infty_0(U) \) such that

1. \( f_1 \geq 0 \),
2. \( \text{supp } f_i \subseteq U_\alpha \), for some \( \alpha \),
3. For every \( p \in U \), there exists a neighborhood \( U_p \) of \( p \) such that \( U_p \cup \text{supp } f_i = \emptyset \) for all \( i > N_p \),
4. \( \sum f_i = 1 \).

**Remark.** Property (4) makes sense because of property (3), because at each point it is a finite sum.

**Remark.** A set of functions satisfying property (4) is called a partition of unity.

**Remark.** Property (2) can be restated as “the partition of unity is subordinate to the cover \( \{U_\alpha : \alpha \in I\} \).”

Let us look at some typical applications of partitions of unity.

The first application is to improper integrals. Let \( \phi : U \to \mathbb{R} \) be a continuous map, and suppose
\[
\int_U \phi \tag{3.170}
\]
is well-defined. Take a partition of unity \( \sum f_i = 1 \). The function \( f_i \phi \) is continuous and compactly supported, so it bounded. Let \( \text{supp } f_i \subseteq Q_i \) for some rectangle \( Q_i \). Then,
\[
\int_{Q_i} f_i \phi \tag{3.171}
\]
is a well-defined R. integral. It follows that
\[
\int_U f_i \phi = \int_{Q_i} f_i \phi. \tag{3.172}
\]

It follows that
\[
\int_U \phi = \sum_{i=1}^\infty \int_{Q_i} f_i \phi. \tag{3.173}
\]

This is proved in Munkres.

The second application of partitions of unity involves cut-off functions.

Let \( f_i \in C^\infty_0(U) \), \( i = 1, 2, 3, \ldots \) be a partition of unity, and let \( A \subseteq U \) be compact.
Lemma 3.31. There exists a neighborhood $U'$ of $A$ in $U$ and a number $N > 0$ such that $A \cup \text{supp } f_i = \phi$ for all $i > N$.

Proof. For any $p \in A$, there exists a neighborhood $U_p$ of $p$ and a number $N_p$ such that $U' \cup \text{supp } f_i = \phi$ for all $i > N_p$. The collection of all these $U_p$ is a cover of $A$. By the H-B Theorem, there exists a finite subcover $U_{p_i}, \ i = 1, 2, 3, \ldots$ of $A$. Take $U_p = \cup U_{p_i}$ and take $N = \max\{N_p\}$.

We use this lemma to prove the following theorem.

Theorem 3.32. Let $A \subseteq \mathbb{R}^n$ be compact, and let $U$ be an open set containing $A$. There exists a function $f \in C_0^\infty(U)$ such that $f \equiv 1$ (identically equal to 1) on a neighborhood $U' \subset U$ of $A$.

Proof. Choose $U'$ and $N$ as in the lemma, and let

$$f = \sum_{i=1}^{N} f_i.$$  \hfill (3.174)

Then $\text{supp } f_i \cap U' = \phi$ for all $i > N$. So, on $U'$,

$$f = \sum_{i=1}^{\infty} f_i = 1.$$  \hfill (3.175)

Such an $f$ can be used to create cut-off functions. We look at an application.

Let $\phi : U \to \mathbb{R}$ be a continuous function. Define $\psi = f\phi$. The new function $\psi$ is called a cut-off function, and it is compactly supported with $\text{supp } \phi \subseteq U$. We can extend the domain of $\psi$ by defining $\psi = 0$ outside of $U$. The extended function $\psi : \mathbb{R}^n \to \mathbb{R}$ is still continuous, and it equals $\phi$ on a neighborhood of $A$.

We look at another application, this time to exhaustion functions.

Definition 3.33. Given an open set $U$, and a collection of compact subsets $A_i, i = 1, 2, 3, \ldots$ of $U$, the sets $A_i$ form an exhaustion of $U$ if $A_i \subseteq \text{Int } A_{i+1}$ and $\cup A_i = U$ (this is just a quick reminder of the definition of exhaustion).

Definition 3.34. A function $\phi \in C^\infty(U)$ is an exhaustion function if

1. $\phi > 0$,
2. the sets $A_i = \phi^{-1}([0,1])$ are compact.

Note that this implies that the $A_i$’s are an exhaustion.
We use the fact that we can always find a partition of unity to show that we can always find exhaustion functions.

Take a partition of unity \( f_i \in C^\infty(U) \), and define

\[
\phi = \sum_{i=1}^{\infty} i f_i. \tag{3.176}
\]

This sum converges because only finitely many terms are nonzero.

Consider any point

\[
p \notin \bigcup_{j \leq i} \text{supp } f_j. \tag{3.177}
\]

Then,

\[
1 = \sum_{k=1}^{\infty} f_k(p) = \sum_{k> i} f_k(p), \tag{3.178}
\]

so

\[
\sum_{\ell=1}^{\infty} \ell f_\ell(p) = \sum_{\ell> i} \ell f_\ell \\
\geq i \sum_{\ell> i} f_\ell \\
= i.
\]

That is, if \( p \notin \bigcup_{j \leq i} \text{supp } f_j \), then \( f(p) > i \). So,

\[
\phi^{-1}([0, i]) \subseteq \bigcup_{j \leq i} \text{supp } f_j, \tag{3.180}
\]

which you should check yourself. The compactness of the r.h.s. implies the compactness of the l.h.s.

Now we look at problem number 4 in section 16 of Munkres. Let \( A \) be an arbitrary subset of \( \mathbb{R}^n \), and let \( g : A \to \mathbb{R}^k \) be a map.

**Definition 3.35.** The function \( g \) is \( C^k \) on \( A \) if for every \( p \in A \), there exists a neighborhood \( U_p \) of \( p \) in \( \mathbb{R}^n \) and a \( C^k \) map \( g^p : U_p \to \mathbb{R}^k \) such that \( g^p|U_p \cap A = g|U_p \cap A \).

**Theorem 3.36.** If \( g : A \to \mathbb{R}^k \) is \( C^k \), then there exists a neighborhood \( U \) of \( A \) in \( \mathbb{R}^n \) and a \( C^k \) map \( \tilde{g} : U \to \mathbb{R}^k \) such that \( \tilde{g} = g \) on \( A \).

**Proof.** This is a very nice application of partition of unity. Read Munkres for the proof. \( \square \)