Lecture 18

We begin with a quick review of permutations (from last lecture).
A permutation of order \(k\) is a bijective map \(\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}\). We denote by \(S_k\) the set of permutations of order \(k\).

The set \(S_k\) has some nice properties. If \(\sigma \in S_k\), then \(\sigma^{-1} \in S_k\). The inverse permutation \(\sigma^{-1}\) is defined by \(\sigma^{-1}(j) = i\) if \(\sigma(i) = j\). Another nice property is that if \(\sigma, \tau \in S_k\), then \(\sigma \tau \in S_k\), where \(\sigma \tau(i) = \sigma(\tau(i))\). That is, if \(\tau(i) = j\) and \(\sigma(j) = k\), then \(\sigma \tau(i) = k\).

Take \(1 \leq i < j \leq k\), and define

\[
\begin{align*}
\tau_{i,j}(i) &= j \\
\tau_{i,j}(j) &= i \\
\tau_{i,j}(\ell) &= \ell, \ell \neq i, j.
\end{align*}
\]

The permutation \(\tau_{i,j}\) is a transposition. It is an elementary transposition of \(j = i + 1\).

Last time we stated the following theorem.

**Theorem 4.13.** Every permutation \(\sigma\) can be written as a product

\[
\sigma = \tau_1 \tau_2 \cdots \tau_r,
\]

where the \(\tau_i\)'s are elementary transpositions.

In the above, we removed the symbol \(\circ\) denoting composition of permutations, but the composition is still implied.

Last time we also defined the sign of a permutation

**Definition 4.14.** The sign of a permutation \(\sigma\) is \((-1)^\sigma = (-1)^r\), where \(r\) is as in the above theorem.

**Theorem 4.15.** The above definition of sign is well-defined, and

\[
(-1)^{\sigma \tau} = (-1)^\sigma (-1)^\tau.
\]

All of the above is discussed in the Multi-linear Algebra Notes.
Part of today's homework is to show the following two statements:

1. \(|S_k| = k!\). The proof is by induction.
2. \((-1)^{\tau_{i,j}} = -1\). Hint: use induction and \(\tau_{i,j} = (\tau_{j-1,j})(\tau_{i,j-1})(\tau_{j-1,j})\), with \(i < j\).

We now move back to the study of tensors. Let \(V\) be an \(n\)-dimensional vector space. We define

\[
V^k = \underbrace{V \times \cdots \times V}_{\text{k factors}}.
\]
We define $L^k(v)$ to be the space of all $k$-linear functions $T : V^k \to \mathbb{R}$. If $T_i \in L^{k_i}$, $i = 1, 2$, and $k = k_1 + k_2$, then $T_1 \otimes T_2 \in L^k$. Decomposable $k$-tensors are of the form $T = \ell_1 \otimes \cdots \otimes \ell_k$, where each $\ell_i \in L^1 = V^*$. Note that $\ell_1 \otimes \cdots \otimes \ell_k(v_1, \ldots, v_k) = \ell_1(v_1) \cdots \ell_k(v_k)$.

We define a permutation operation on tensors. Take $\sigma \in S_k$ and $T \in L^k(V)$.

**Definition 4.16.** We define the map $T^\sigma : V^k \to \mathbb{R}$ by

$$T^\sigma(v_1, \ldots, v_k) = T(v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(k)}).$$ (4.46)

Clearly, $T^\sigma \in L^k(V)$. We have the following useful formula:

**Claim.**

$$(T^\sigma)^\tau = T^{\tau\sigma}. $$ (4.47)

**Proof.**

$$T^{\tau\sigma}(v_1, \ldots, v_k) = T(v_{\tau^{-1}(1)}v_{\sigma^{-1}(1)}, \ldots, v_{\tau^{-1}(k)}v_{\sigma^{-1}(k)})
= T^\sigma(v_{\tau^{-1}(1)}, \ldots, v_{\tau^{-1}(k)})
= (T^\sigma)^\tau(v_1, \ldots, v_k).$$ (4.48)

Let us look at what the permutation operation does to a decomposable tensor $T = \ell_1 \otimes \cdots \otimes \ell_k$.

$$T^\sigma(v_1, \ldots, v_k) = \ell_1(v_{\sigma^{-1}(1)}) \cdots \ell_k(v_{\sigma^{-1}(k)}).$$ (4.49)

The $i$th factor has the subscript $\sigma^{-1}(i) = j$, where $\sigma(j) = i$, so the the $i$th factor is $\ell_{\sigma(j)}(v_j)$. So

$$T^\sigma(v_1, \ldots, v_k) = \ell_{\sigma(1)}(v_1) \cdots \ell_{\sigma(k)}(v_k)
= (\ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)})(v_1, \ldots, v_k).$$ (4.50)

To summarize,

$$\begin{cases} T = \ell_1 \otimes \cdots \otimes \ell_k \\ T^\sigma = \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}. \end{cases}$$ (4.51)

**Proposition 4.17.** The mapping $T \in L^k \to T^\sigma \in L^k$ is linear.

We leave the proof of this as an exercise.

**Definition 4.18.** A tensor $T \in L^k(V)$ is alternating if $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$.

**Definition 4.19.** We define

$$\mathcal{A}^k(V) = \text{the set of all alternating } k\text{-tensors.}$$ (4.52)
By our previous claim, $\mathcal{A}^k$ is a vector space.
The alternating operator $\text{Alt}$ can be used to create alternating tensors.

**Definition 4.20.** Given a $k$-tensor $T \in \mathcal{L}^k(V)$, we define the alternating operator $\text{Alt} : \mathcal{L}^k(V) \rightarrow \mathcal{A}^k(V)$ by

$$\text{Alt} (T) = \sum_{\tau \in S_k} (-1)^{\tau} T^{\tau}. \quad (4.53)$$

**Claim.** The alternating operator has the following properties:

1. $\text{Alt} (T) \in \mathcal{A}^k(V)$,
2. If $T \in \mathcal{A}^k(V)$, then $\text{Alt} (T) = k! T$,
3. $\text{Alt} (T^\sigma) = (-1)^{\sigma} \text{Alt} (T)$,
4. The map $\text{Alt} : \mathcal{L}^k(V) \rightarrow \mathcal{A}^k(V)$ is linear.

**Proof.**

1. $\text{Alt} (T) = \sum_{\tau} (-1)^{\tau} T^{\tau}, \quad (4.54)$

so

$$\text{Alt} (T)^\sigma = \sum_{\tau} (-1)^{\tau} (T^{\tau})^{\sigma}$$

$$= \sum_{\tau} (-1)^{\tau} T^{\sigma \tau}$$

$$= (-1)^{\sigma} \sum_{\sigma \tau} (-1)^{\tau} T^{\sigma \tau}$$

$$= (-1)^{\sigma} \text{Alt} (T). \quad (4.55)$$

2. $\text{Alt} (T) = \sum_{\tau} (-1)^{\tau} T^{\tau}, \quad (4.56)$

but $T^{\tau} = (-1)^{\tau} T$, since $T \in \mathcal{A}^k(V)$. So

$$\text{Alt} (T) = \sum_{\tau} (-1)^{\tau} (-1)^{\tau} T$$

$$= k! T. \quad (4.57)$$

3. $\text{Alt} (T^\sigma) = \sum_{\tau} (-1)^{\tau} (T^\sigma)^{\tau}$

$$= \sum_{\tau} (-1)^{\tau} T^{\tau \sigma}$$

$$= (-1)^{\sigma} \sum_{\tau \sigma} (-1)^{\tau \sigma} T^{\tau \sigma}$$

$$= (-1)^{\sigma} \text{Alt} (T). \quad (4.58)$$
4. We leave the proof as an exercise.

Now we ask ourselves: what is the dimension of $\mathcal{A}^k(V)$? To answer this, it is best to write a basis.

Earlier we found a basis for $\mathcal{L}^k$. We defined $e_1, \ldots, e_n$ to be a basis of $V$ and $e_1^*, \ldots, e_n^*$ to be a basis of $V^*$. We then considered multi-indices $I = (i_1, \ldots, i_k)$, $1 \leq i_r \leq n$ and defined \{e_I^* = e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*, I \text{ a multi-index}\} to be a basis of $\mathcal{L}^k$. For any multi-index $J = (j_1, \ldots, j_k)$, we had

$$e_I^*(e_{j_1}, \ldots, e_{j_k}) = \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases} \quad (4.59)$$

**Definition 4.21.** A multi-index $I = (i_1, \ldots, i_k)$ is repeating if $i_r = i_s$ for some $r < s$.

**Definition 4.22.** The multi-index $I$ is strictly increasing if $1 \leq i_1 < \cdots < i_k \leq n$.

**Notation.** Given $\sigma \in S_k$ and $I = (i_1, \ldots, i_k)$, we denote $I^\sigma = (i_{\sigma(1)}, \ldots, i_{\sigma(k)})$.

**Remark.** If $J$ is a non-repeating multi-index, then there exists a permutation $\sigma$ such that $J = I^\sigma$, where $I$ is strictly increasing.

$$e_J^* = e_I^* = e_{\sigma(i_1)}^* \otimes \cdots \otimes e_{\sigma(i_k)}^* = (e_{I^\sigma})^\sigma. \quad (4.60)$$

Define $\psi_I = \text{Alt} (e_I^*)$.

**Theorem 4.23.**

1. $\psi_{I^\sigma} = (-1)^\sigma \psi_I$,

2. If $I$ is repeating, then $\psi_I = 0$,

3. If $I, J$ are strictly increasing, then

$$\psi_I(e_{j_1}, \ldots, e_{j_k}) = \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases} \quad (4.61)$$

**Proof.**

1.

$$\psi_{I^\sigma} = \text{Alt} e_{I^\sigma}^*$$

$$= \text{Alt} ((e_I^*)^\sigma)$$

$$= (-1)^\sigma \text{Alt} e_I^*$$

$$= (-1)^\sigma \psi_I. \quad (4.62)$$

2. Suppose that $I$ is repeating. Then $I = I^\tau$ for some transposition $\tau$. So $\psi_I = (-1)^\tau \psi_I$. But (as you proved in the homework) $(-1)^\tau = -1$, so $\psi_I = 0$. 


3. 

\[ \psi_I = \text{Alt} (e_I^*) \]
\[ = \sum_{\tau} (-1)^{\tau} e_{I^\tau}^*, \quad (4.63) \]

so

\[ \psi_I(e_{j_1}, \ldots, e_{j_k}) = \sum_{\tau} (-1)^{\tau} e_{I^\tau}(e_{j_1}, \ldots, e_{j_k}) \quad (4.64) \]

\[ \begin{cases} 1 & \text{if } I^\tau = J, \\ 0 & \text{if } I^\tau \neq J. \end{cases} \]

But \( I^\tau = J \) only if \( \tau \) is the identity permutation (because both \( I^\tau \) and \( J \) are strictly increasing). The only non-zero term in the sum is when \( \tau \) is the identity permutation, so

\[ \psi_I(e_{j_1}, \ldots, e_{j_k}) = \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases} \quad (4.65) \]

\[ \square \]

**Corollary 5.** The alternating \( k \)-tensors \( \psi_I \), where \( I \) is strictly increasing, are a basis of \( \mathcal{A}^k(V) \).

**Proof.** Take \( T \in \mathcal{A}^k(V) \). The tensor \( T \) can be expanded as \( T = \sum c_I e_I^* \). So

\[ \text{Alt} (T) = k! \sum c_I \text{Alt} (e_I^*) \]
\[ = k! \sum c_I \psi_I. \quad (4.66) \]

If \( I \) is repeating, then \( \psi_I = 0 \). If \( I \) is non-repeating, then \( I = J^\sigma \), where \( J \) is strictly increasing. Then \( \psi_I = (-1)^\sigma \psi_J \).

So, we can replace all multi-indices in the sum by strictly increasing multi-indices,

\[ T = \sum a_I \psi_I, \quad I \text{'s strictly increasing.} \quad (4.67) \]

Therefore, the \( \psi_I \)'s span \( \mathcal{A}^k(V) \). Moreover, the \( \psi_I \)'s are a basis if and only if the \( a_i \)'s are unique. We show that the \( a_I \)'s are unique.

Let \( J \) be any strictly increasing multi-index. Then

\[ T(e_{j_1}, \ldots, e_{j_k}) = \sum a_I \psi_I(e_{j_1}, \ldots, e_{j_k}) \]
\[ = a_J, \quad (4.68) \]

by property (3) of the previous theorem. Therefore, the \( \psi_I \)'s are a basis of \( \mathcal{A}^k(V) \). \[ \square \]