Lecture 2

1.6 Compactness

As usual, throughout this section we let \((X, d)\) be a metric space. We also remind you from last lecture we defined the open set

\[
U(x_0, \lambda) = \{x \in X : d(x, x_0) < \lambda\}. \tag{1.10}
\]

**Remark.** If \(U(x_0, \lambda) \subseteq U(x_1, \lambda_1)\), then \(\lambda_1 > d(x_0, x_1)\).

**Remark.** If \(A_i \subseteq U(x_0, \lambda_i)\) for \(i = 1, 2\), then \(A_1 \cup A_2 \subseteq U(x_0, \lambda_1 + \lambda_2)\).

Before we define compactness, we first define the notions of boundedness and covering.

**Definition 1.19.** A subset \(A\) of \(X\) is **bounded** if \(A \subseteq U(x_0, \lambda)\) for some \(\lambda\).

**Definition 1.20.** Let \(A \subseteq X\). A collection of subsets \(\{U_\alpha \subseteq X, \alpha \in I\}\) is a **cover** of \(A\) if

\[
A \subseteq \bigcup_{\alpha \in I} U_\alpha.
\]

Now we turn to the notion of compactness. First, we only consider compact sets as subsets of \(\mathbb{R}^n\).

For any subset \(A \subseteq \mathbb{R}^n\),

\[
A \text{ is compact} \iff A \text{ is closed and bounded}.
\]

The above statement holds true for \(\mathbb{R}^n\) but not for general metric spaces. To motivate the definition of compactness for the general case, we give the Heine-Borel Theorem.

**Heine-Borel (H-B) Theorem.** Let \(A \subseteq \mathbb{R}^n\) be compact and let \(\{U_\alpha, \alpha \in I\}\) be a cover of \(A\) by open sets. Then a finite number of \(U_\alpha\)'s already cover \(A\).

The property that a finite number of the \(U_\alpha\)'s cover \(A\) is called the Heine-Borel (H-B) property. So, the H-B Theorem can be restated as follows: If \(A\) is compact in \(\mathbb{R}^n\), then \(A\) has the H-B property.

**Sketch of Proof.** First, we check the H-B Theorem for some simple compact subsets of \(\mathbb{R}^n\). Consider rectangles \(Q = I_1 \times \cdots \times I_n \subset \mathbb{R}^n\), where \(I_k = [a_k, b_k]\) for each \(k\). Starting with one dimension, it can by shown by induction that these rectangles have the H-B property.

To prove the H-B theorem for general compact subsets, consider any closed and bounded (and therefore compact) subset \(A\) of \(\mathbb{R}^n\). Since \(A\) is bounded, there exists a rectangle \(Q\) such that \(A \subseteq Q\). Suppose that the collection of subsets \(\{U_\alpha, \alpha \in I\}\) is
an open cover of $A$. Then, define $U_o = \mathbb{R}^n - A$ and include $U_o$ in the open cover. The rectangle $Q$ has the H-B property and is covered by this new cover, so there exists a finite subcover covering $Q$. Furthermore, the rectangle $Q$ contains $A$, so the finite subcover also covers $A$, proving the H-B Theorem for general compact subsets.

The following theorem further motivates the general definition for compactness.

**Theorem 1.21.** If $A \subseteq \mathbb{R}^n$ has the H-B property, then $A$ is compact.

**Sketch of Proof.** We need to show that the H-B property implies $A$ is bounded (which we leave as an exercise) and closed (which we prove here).

To show that $A$ is closed, it is sufficient to show that $A^c$ is open. Take any $x_o \in A^c$, and define

$$C_N = \{x \in \mathbb{R}^n : d(x, x_o) \leq 1/N\},$$

(1.11) and

$$U_N = C_N^c.$$ 

(1.12)

Then,

$$\bigcap C_N = \{x_o\}$$

(1.13)

and

$$\bigcup U_N = \mathbb{R}^n - \{x_o\}.$$  

(1.14)

The $U_N$'s cover $A$, so the H-B Theorem implies that there is a finite subcover $\{U_{N_1}, \ldots, U_{N_k}\}$ of $A$. We can take $N_1 < N_2 < \cdots < N_k$, so that $A \subseteq U_{N_k}$. By taking the complement, it follows that $C_{N_k} \subseteq A^c$. But $U(x_o, 1/N_k) \subseteq C_{N_k}$, so $x_o$ is contained in an open subset of $A^c$. The above holds for any $x_o \in A^c$, so $A^c$ is open.

Let us consider the above theorem for arbitrary metric space $(X, d)$ and $A \subseteq X$.

**Theorem 1.22.** If $A \subseteq X$ has the H-B property, then $A$ is closed and bounded.

**Sketch of Proof.** The proof is basically the same as for the previous theorem.

Unfortunately, the converse is not always true. Finally, we come to our general definition of compactness.

**Definition 1.23.** A subset $A \subseteq X$ is *compact* if it has the H-B property.

Compact sets have many useful properties, some of which we list here in the theorems that follow.

**Theorem 1.24.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces, and let $f : X \rightarrow Y$ be a continuous map. If $A$ is a compact subset of $X$, then $f(A)$ is a compact subset of $Y$. 


Proof. Let \( \{U_\alpha, \alpha \in I\} \) be an open covering of \( f(A) \). Each pre-image \( f^{-1}(U_\alpha) \) is open in \( X \), so \( \{f^{-1}(U_\alpha) : \alpha \in I\} \) is an open covering of \( A \). The H-B Theorem says that there is a finite subcover \( \{f^{-1}(U_\alpha) : 1 \leq i \leq N\} \). It follows that the collection \( \{U_\alpha : 1 \leq i \leq N\} \) covers \( f(A) \), so \( f(A) \) is compact. \( \square \)

A special case of the above theorem proves the following theorem.

**Theorem 1.25.** Let \( A \) be a compact subset of \( X \) and \( f : X \rightarrow \mathbb{R} \) be a continuous map. Then \( f \) has a maximum point on \( A \).

**Proof.** By the above theorem, \( f(A) \) is compact, which implies that \( f(a) \) is closed and bounded. Let \( a = \text{l.u.b.} f(A) \). The point \( a \) is in \( f(A) \) because \( f(A) \) is closed, so there exists an \( x_o \in A \) such that \( f(x_o) = a \). \( \square \)

Another useful property of compact sets involves the notion of uniform continuity.

**Definition 1.26.** Let \( f : X \rightarrow \mathbb{R} \) be a continuous function, and let \( A \) be a subset of \( X \). The map \( f \) is uniformly continuous on \( A \) if for every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that
\[
d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon,
\]
for all \( x, y \in A \).

**Theorem 1.27.** If \( f : X \rightarrow Y \) is continuous and \( A \) is a compact subset of \( X \), then \( f \) is uniformly continuous on \( A \).

**Proof.** Let \( p \in A \). There exists a \( \delta_p > 0 \) such that \( |f(x) - f(p)| < \epsilon/2 \) for all \( x \in U(p, \delta_p) \). Now, consider the collection of sets \( \{U(p, \delta_p/2) : p \in A\} \), which is an open cover of \( A \). The H-B Theorem says that there is a finite subcover \( \{U(p_i, \delta_{p_i}/2) : 1 \leq i \leq N\} \). Choose \( \delta \leq \min \delta_{p_i}/2 \). The following claim finishes the proof.

**Claim.** If \( d(x, y) < \delta \), then \( |f(x) - f(y)| < \epsilon \).

**Proof.** Given \( x \), choose \( p_i \) such that \( x \in U(p_i, \delta_{p_i}/2) \). So, \( d(p_i, x) < \delta_{p_i}/2 \) and \( d(x, y) < \delta < \delta_{p_i}/2 \). By the triangle inequality we conclude that \( d(p_i, y) < \delta_{p_i} \). This shows that \( x,y \in U(p_i, \delta_{p_i}) \), which implies that \( |f(x) - f(p_i)| < \epsilon/2 \) and \( |f(y) - f(p_i)| < \epsilon/2 \).

Finally, by the triangle inequality, \( |f(x) - f(y)| < \epsilon \), which proves our claim. \( \square \)

### 1.7 Connectedness

As usual, let \((X, d)\) be a metric space.

**Definition 1.28.** The metric space \((X, d)\) is connected if it is impossible to write \( X \) as a disjoint union \( X = U_1 \cup U_2 \) of non-empty open sets \( U_1 \) and \( U_2 \).
Note that disjoint simply means that \( U_1 \cap U_2 = \emptyset \), where \( \emptyset \) is the empty set.

A few simple examples of connected spaces are \( \mathbb{R}, \mathbb{R}^n \), and \( I = [a, b] \). The following theorem shows that a connected space gets mapped to a connected subspace by a continuous function.

**Theorem 1.29.** Given metric spaces \((X, d_X)\) and \((Y, d_Y)\), and a continuous map \( f : X \to Y \), it follows that

\[
X \text{ is connected} \implies f(X) \text{ is connected}.
\]

**Proof.** Suppose \( f(X) \) can be written as a union of open sets \( f(X) = U_1 \cup U_2 \) such that \( U_1 \cap U_2 = \emptyset \). Then \( X = f^{-1}(U_1) \cup f^{-1}(U_2) \) is a disjoint union of open sets. This contradicts that \( X \) is connected. \( \square \)

The intermediate-value theorem follows as a special case of the above theorem.

**Intermediate-value Theorem.** Let \((X, d)\) be connected and \( f : X \to \mathbb{R} \) be a continuous map. If \( a, b \in f(X) \) and \( a < r < b \), then \( r \in f(X) \).

**Proof.** Suppose \( r \notin f(X) \). Let \( A = (-\infty, r) \) and \( B = (r, \infty) \). Then \( X = f^{-1}(A) \cup f^{-1}(B) \) is a disjoint union of open sets, a contradiction. \( \square \)