Lecture 23

Let $U$ be an open set in $\mathbb{R}^n$. For each $k = 0, \ldots, n - 1$, we define the differential operator

$$d : \Omega^k(U) \to \Omega^{k+1}(U).$$

(4.175)

These maps are the $n$ basic vector calculus operations in $n$-dimensional calculus. We review how $d$ is defined.

For $k = 0$, $\Omega^0(U) = C^\infty(U)$. Let $f \in C^\infty(U)$, and let $c = f(p)$, where $p \in U$. The mapping $df_p : T_p\mathbb{R}^n \to T_c\mathbb{R} = \mathbb{R}$ maps $T_p\mathbb{R}^n$ to $\mathbb{R}$, so $df_p \in T_c^n\mathbb{R}^n$. The map $df \in \Omega^1(U)$ is a one-form that maps $p \in U$ to $df_p \in T_c^n\mathbb{R}^n$. A formula for this in coordinates is

$$df = \sum \frac{\partial f}{\partial x_i} dx_i.$$  

(4.176)

In $k$ dimensions, $d$ is a map

$$d : \Omega^k(U) \to \Omega^{k+1}(U).$$

(4.177)

Given $\omega \in \Omega^k(U)$, $\omega$ can be written uniquely as

$$\omega = \sum_I a_I dx_I = \sum_I a_I dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

(4.178)

where $i_1 < \cdots < i_k$ and each $a_I \in C^\infty(U)$. Then, we define

$$d\omega = \sum_I da_I \wedge dx_I = \sum_{i,I} \frac{\partial a_I}{\partial x_i} dx_i \wedge dx_I,$$

(4.179)

where each $I$ is strictly increasing.

The following are some basic properties of the differential operator $d$:

1. If $\mu \in \Omega^k(U)$ and $\nu \in \Omega^\ell(U)$, then

$$d\mu \wedge \nu = d\mu \wedge \nu + (-1)^k \mu \wedge d\nu.$$  

(4.180)

2. For and $\omega \in \Omega^k(U)$,

$$d(d\omega) = 0.$$  

(4.181)

Remark. Let $I$ be any multi-index, and let $a_I \in C^\infty(U)$. Then

$$d(a_I dx_I) = da_I \wedge dx_I.$$  

(4.182)
We now prove the above two basic properties of the differential operator.

**Claim.** If $\mu \in \Omega^k(U)$ and $\nu \in \Omega^l(U)$, then

$$d\mu \wedge \nu = d\mu \wedge \nu + (-1)^k \mu \wedge d\nu. \quad (4.183)$$

**Proof.** Take $\mu = \sum a_I dx_I$ and $\nu = \sum b_J dx_J$, where $I, J$ are strictly increasing. Then

$$\mu \wedge \nu = \sum a_I b_J \underbrace{dx_I \wedge dx_J}_{\text{no longer increasing}}. \quad (4.184)$$

Thus

$$d(\mu \wedge \nu) = \sum_{i,I,J} \frac{\partial a_I}{\partial x_i} b_J dx_I \wedge dx_J \wedge dx_J$$

$$= \sum_{i,I} \frac{\partial a_I}{\partial x_i} b_J dx_I \wedge dx_J \wedge dx_J \quad (I) \quad (4.185)$$

$$+ \sum_i a_I \frac{\partial b_J}{\partial x_i} dx_I \wedge dx_I \wedge dx_J, \quad (II)$$

We calculate sums (I) and (II) separately.

$$(I) = \sum_{i,I,J} \frac{\partial a_I}{\partial x_i} dx_I \wedge dx_J \wedge dx_J$$

$$= \left( \sum_{i,I} \frac{\partial a_I}{\partial x_i} dx_I \wedge dx_J \right) \wedge \sum_{J} b_J dx_J$$

$$= d\mu \wedge \nu. \quad (4.186)$$

$$(II) = \sum_{i,I,J} a_I \frac{\partial b_J}{\partial x_i} dx_I \wedge dx_J \wedge dx_J$$

$$= (-1)^k \sum_{i,I,J} a_I dx_I \wedge \frac{\partial b_J}{\partial x_i} dx_I \wedge dx_J$$

$$= (-1)^k \left( \sum_I a_I dx_I \right) \wedge \sum_{i,J} \frac{\partial b_J}{\partial x_i} dx_I \wedge dx_J$$

$$= (-1)^k \mu \wedge d\nu. \quad (4.187)$$

So,

$$d(\mu \wedge \nu) = (I) + (II)$$

$$= d\mu \wedge \nu + (-1)^k \mu \wedge d\nu. \quad (4.188)$$

$\square$
Claim. For and $\omega \in \Omega^k(U)$,  
\[ d(d\omega) = 0. \] (4.189)

Proof. Let $\omega = \sum a_I dx_I$, so  
\[ d\omega = \sum_{j,I} \frac{\partial a_I}{\partial x_j} dx_j \wedge dx_I. \] (4.190)

Then,  
\[ d(d\omega) = \sum_{i,j,I} \frac{\partial^2 a_I}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_I. \] (4.191)

Note that if $i = j$, then there is a repeated term in the wedge product, so  
\[ d(d\omega) = \sum_{i<j} \frac{\partial^2 a_I}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_I \] (4.192)
\[ + \sum_{i>j} \frac{\partial^2 a_I}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_I. \] (4.193)

Note that $dx_i \wedge dx_j = -dx_j \wedge dx_i$. We relabel the second summand to obtain  
\[ d(d\omega) = \sum_{i<j} \left( \frac{\partial^2 a_I}{\partial x_i \partial x_j} - \frac{\partial^2 a_I}{\partial x_j \partial x_i} \right) dx_i \wedge dx_j \wedge dx_I \] (4.194)
\[ = 0. \]

\[ \square \]

Definition 4.42. A $k$-form $\omega \in \Omega^k(U)$ is decomposable if $\omega = \mu_1 \wedge \cdots \wedge \mu_k$, where each $\mu_i \in \Omega^1(U)$.

Theorem 4.43. If $\omega$ is decomposable, then  
\[ d\omega = \sum_{i=1}^{k} (-1)^{i-1} \mu_1 \wedge \cdots \wedge \mu_{i-1} \wedge d\mu_i \wedge \mu_{i+1} \wedge \cdots \wedge \mu_k. \] (4.195)

Proof. The proof is by induction.
The case \( k = 1 \) is obvious. We show that if the theorem is true for \( k - 1 \), then the theorem is true for \( k \).

\[
d((\mu_1 \wedge \cdots \wedge \mu_{k-1}) \wedge \mu_k) = (d(\mu_1 \wedge \cdots \wedge \mu_{k-1})) \wedge \mu_k \\
+ (-1)^{k-1}(\mu_1 \wedge \cdots \wedge \mu_{k-1}) \wedge d\mu_k \\
= \sum_{i=1}^{k-1}(-1)^{i-1}\mu_1 \wedge \cdots \wedge d\mu_i \wedge \cdots \wedge \mu_{k-1} \wedge \mu_k \\
+ (-1)^{k-1}(\mu_1 \wedge \cdots \wedge \mu_{k-1} \wedge \mu_k) \\
= \sum_{i=1}^{k}(-1)^{i-1}\mu_1 \wedge \cdots \wedge d\mu_i \wedge \cdots \wedge \mu_k.
\]

\( \Box \)

4.10 Pullback Operation on Exterior Forms

Another important operation in the theory of exterior forms is the pullback operator. This operation is not introduced in 18.01 or 18.02, because vector calculus in not usually taught rigorously.

Let \( U \) be open in \( \mathbb{R}^n \) and \( V \) be open in \( \mathbb{R}^m \), and let \( f : U \to V \) be a \( C^\infty \) map. We can write out in components \( f = (f_1, \ldots, f_n) \), where each \( f_i \in C^\infty(U) \). Let \( p \in U \) and \( q = f(p) \).

The pullback of the map \( df_p : T_p\mathbb{R}^m \to T_q\mathbb{R}^n \) is

\[
(df_p)^* : \Lambda^k(T_q^*\mathbb{R}^n) \to \Lambda^k(T_p^*\mathbb{R}^m).
\]

(4.197)

Suppose you have a \( k \)-form \( \omega \) on \( V \).

\[
\omega \in \Omega^k(V),
\]

(4.198)

\[
\omega_q \in \Lambda^k(T_q^*\mathbb{R}^n).
\]

(4.199)

Then

\[
(df_p)^*\omega_q \in \Lambda^k(T_p^*\mathbb{R}^m).
\]

(4.200)

**Definition 4.44.** \( f^*\omega \) is the \( k \)-from whose value at \( p \in U \) is \((df_p)^*\omega_q\).

We consider two examples. Suppose \( \phi \in \Omega^0(V) = C^\infty(V) \). Then \( f^*\phi(p) = \phi(q) \), so \( f^*\phi = \phi \circ f \), where \( f : U \to V \) and \( \phi : V \to \mathbb{R} \).

Again, suppose that \( \phi \in \Omega^0(V) = C^\infty(V) \). What is \( f^*d\phi \)? Let \( f(p) = q \). We have the map \( d\phi_q : T_p\mathbb{R}^n \to T_c\mathbb{R} = \mathbb{R} \), where \( c = \phi(q) \). So,

\[
(df_p)^*(d\phi)_q = d\phi_q \circ df_p \\
= d(\phi \circ f)_p.
\]

(4.201)
Therefore, 
\[ f^*d\phi = df^*\phi. \] (4.202)

Suppose that \( \mu \in \Omega^k(V) \) and \( \nu \in \Omega^l(V) \). Then
\[ (f^*(\mu \wedge \nu))_p = (df^*_p)^*(\mu_q \wedge \nu_q) \]
\[ = (df^*_p)^*\mu_q \wedge (df^*_p)^*\nu_q. \] (4.203)

Hence,
\[ f^*(\mu \wedge \nu) = f^*\mu \wedge f^*\nu. \] (4.204)

We now obtain a coordinate formula for \( f^* \).

Take \( \omega \in \Omega^k(V) \). We can write \( \omega = \sum a_I dx_i \wedge \cdots \wedge dx_{ik} \), where each \( a_I \in C^\infty(U) \).

Then
\[ f^*\omega = \sum f^*a_I f^*dx_i \wedge \cdots \wedge f^*dx_{ik} \]
\[ = \sum f^*a_I df_i \wedge \cdots \wedge df_{ik}, \] (4.205)

where we used the result that \( f^*dx_i = dx_i \circ f = df_i \).

Note that \( df_i = \sum \frac{\partial f_i}{\partial x_j} dx_j \), where \( \frac{\partial f_i}{\partial x_j} \in C^\infty(U) \). Also, \( f^*a_I = a_I \circ f \in C^\infty(U) \),

which shows that
\[ f^*\omega \in \Omega^k(U). \] (4.206)

The following theorem states a very useful property of the pullback operator.

**Theorem 4.45.** Let \( \omega \in \Omega^k(V) \). Then,
\[ df^*\omega = f^*d\omega. \] (4.207)

**Proof.** We have already checked this for \( \omega = \phi \in C^\infty(V), k = 0 \) already. We now prove the general case.

We can write \( \omega = \sum a_I dx_I \). Then
\[ f^*\omega = \sum f^*a_I df_i \wedge \cdots \wedge df_{ik}. \] (4.208)

So,
\[ df^*\omega = \sum df^*a_I \wedge df_i \wedge \cdots \wedge df_{ik} \]
\[ + \sum f^*a_I \wedge d(df_i \wedge \cdots \wedge df_{ik}) \] (4.209)

Note that
\[ d(df_i \wedge \cdots \wedge df_{ik}) = \sum_{r=1}^{k} (-1)^{r-1} df_i \wedge \cdots \wedge d(df_{ir}) \wedge \cdots \wedge df_{ik}. \] (4.210)
We know that $d(df_{i_+}) = 0$, so
\[
df^* \omega = \sum_I df^* a_I \wedge df_{i_1} \wedge \cdots \wedge df_{i_k} \\
= \sum_I f^* da_I \wedge f^*(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \\
= f^* (\sum_I da_I \wedge dx_I) \\
= f^* d\omega. \\
\tag{4.211}
\]