Lecture 35

Before moving on to integration, we make a few more remarks about orientations. Let $X, Y$ be oriented manifolds. A diffeomorphism $f : X \to Y$ is orientation preserving if for every $p \in X$, the map

$$df_p : T_p X \to T_q Y$$

is orientation preserving, where $q = f(p)$.

Let $V$ be open in $X$, let $U$ be open in $\mathbb{R}^n$, and let $\phi : U \to V$ be a parameterization.

**Definition 6.32.** The map $\phi$ is an oriented parameterization if it is orientation preserving.

Suppose $\phi$ is orientation reversing. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be the linear map defined by

$$A(x_1, \ldots, x_n) = (-x_1, x_2, \ldots, x_n).$$

The map $A$ is orientation reversing. Let $U' = A^{-1}(U)$, and define $\phi' = \phi \circ A : U' \to V$. Both $\phi$ and $A$ are orientation reversing, so $\phi'$ is orientation preserving.

Thus, for every point $p \in X$, there exists an oriented parameterization of $X$ at $p$.

### 6.7 Integration on Manifolds

Our goal for today is to take any $\omega \in \Omega^n_c(X)$ and define

$$\int_X \omega.$$  

First, we consider a special case:

Let $\phi : U \to V$ be an oriented parameterization. Let $U$ be open in $\mathbb{R}^n$, and let $V$ be open in $X$. Take any $\omega \in \Omega^n_c(V)$. Then

$$\int_V \omega = \int_U \phi^* \omega,$$

where $\phi^* \omega = f(x)dx_1 \wedge \cdots \wedge dx_n$, where $f \in C_0^\infty(U)$ and

$$\int_U \phi^* \omega = \int_U f.$$  

**Claim.** The above definition for $\int \omega$ does not depend on the choice of oriented parameterization $\phi$. 

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Proof. Let \( \phi_i : U_i \to V, \ i = 1, 2, \) be oriented parameterizations. Let \( \omega \in \Omega^n_c(V_1 \cap V_2) \). Define
\[
U_{1,2} = \phi_1^{-1}(V_1 \cap V_2), \\
U_{2,1} = \phi_2^{-1}(V_1 \cap V_2),
\]
which are open sets in \( \mathbb{R}^n \).

Both \( \phi_1 \) and \( \phi_2 \) are diffeomorphisms, and we have the diagram
\[
\begin{array}{ccc}
V_1 \cap V_2 & \xrightarrow{\phi_1} & V_1 \\
\phi_1 & \downarrow & \downarrow \phi_2 \\
U_{1,2} & \xrightarrow{f} & U_{2,1}.
\end{array}
\]
Therefore, \( f = \phi_2^{-1} \circ \phi_1 \) is a diffeomorphism, and \( \phi_1 = \phi_2 \circ f \). Integrating,
\[
\int_{U_1} \phi_1^* \omega = \int_{U_{1,2}} \phi_1^* \omega \\
= \int_{U_{1,2}} (\phi_2 \circ f)^* \omega \\
= \int_{U_{1,2}} f^*(\phi_2^* \omega).
\]
Note that \( f \) is orientation preserving, because \( \phi_1 \) and \( \phi_2 \) are orientation preserving.

Using the change of variables formula,
\[
\int_{U_{1,2}} f^* \phi_2^* \omega = \int_{U_{2,1}} \phi_2^* \omega \\
= \int_{U_2} \phi_2^* \omega.
\]
So, for all \( \omega \in \Omega^n_c(V_1 \cap V_2) \),
\[
\int_{V_1} \omega = \int_{U_1} \phi_1^* \omega = \int_{U_2} \phi_2^* \omega = \int_{V_2} \omega.
\]

Above, we showed above how to take integrals over open sets, and now we generalize.

To define the integral, we need the following two inputs:

1. a set of oriented parameterizations \( \phi_i : U_i \to V_i, \ i = 1, 2, \ldots, \) such that \( X = \bigcup V_i \),
2. a partition of unity $\rho_i \in C^\infty_0 (V_i)$ subordinate to the cover $\{V_i\}$.

**Definition 6.33.** Let $\omega \in \Omega^n_c (X)$. We define the integral

$$\int_X \omega = \sum_{i=1}^{\infty} \int_{V_i} \rho_i \omega.$$  \hspace{1cm} (6.96)

One can check various standard properties of integrals, such as linearity:

$$\int_X (\omega_1 + \omega_2) = \int_X \omega_1 + \int_X \omega_2.$$ \hspace{1cm} (6.97)

We now show that this definition is independent of the choice of the two inputs (the parameterizations and the partition of unity).

Consider two different inputs:

1. oriented parameterizations $\phi^j : U^j \to V^j$, $j = 1, 2, \ldots$, such that $X = \bigcup V^j$,

2. a partition of unity $\rho^j_i \in C^\infty_0 (V^j)$ subordinate to the cover $\{V^j\}$.

Then,

$$\int_{V_i} \rho^j_i \omega = \int_{V_i} \left( \sum_{j=1}^{\infty} \rho^j_i \omega \right)$$

$$= \sum_{j=1}^{\infty} \int_{V_i} \rho^j_i \rho^j_i \omega \hspace{1cm} (6.98)$$

$$= \sum_{j=1}^{\infty} \int_{V_i \cap V^j} \rho^j_i \rho^j_i \omega.$$  

Summing over $i$,

$$\sum_i \int_{V_i} \rho^j_i \omega = \sum_{i,j=1}^{\infty} \int_{V_i \cap V^j} \rho_i \rho^j_i \omega$$

$$= \sum_j \int_{V^j} \rho^j_i \omega,$$ \hspace{1cm} (6.99)

where the first term equals the last term by symmetry. Therefore, the integral $\int \omega$ is independent of the choices of these two inputs.

Let $X \subseteq \mathbb{R}^N$ be an oriented connected $n$-dimensional manifold.

**Theorem 6.34.** For any $\omega \in \Omega^n_c (X)$, the following are equivalent:

1. $\int_X \omega = 0$, 

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2. $\omega \in d\Omega^{n-1}_c(X)$.

**Proof.** This will be a five step proof:

Step 1: The following lemma is called the Connectivity Lemma.

**Lemma 6.35.** Given $p, q \in X$, there exists open sets $W_j$, $j = 0, \ldots, N + 1$, such that each $W_j$ is diffeomorphic to an open set in $\mathbb{R}^n$, and such that $p \in W_0$, $q \in W_{N+1}$, and $W_i \cap W_{i+1} \neq \emptyset$.

**Proof Idea:** Fix $p$. The points $q$ for which this is true form an open set. The points $q$ for which this isn’t true also form an open set. Since $X$ is connected, only one of these sets is in $X$. \hfill \Box

Step 2: Let $\omega_1, \omega_2 \in \Omega^n_c(X)$. We say that $\omega_1 \sim \omega_2$ if

$$\int_X \omega_1 = \int_X \omega_2. \tag{6.100}$$

We can restate the theorem as

$$\omega_1 \sim \omega_2 \iff \omega_1 - \omega_2 \in d\Omega^{n-1}_c(X). \tag{6.101}$$

Step 3: It suffices to prove the statement (6.101) for $\omega_1 \in \Omega^n_c(V)$ and $\omega_2 \in \Omega^n_c(V')$, where $V, V'$ are diffeomorphic to open sets in $\mathbb{R}^n$.

Step 4: We use a partition of unity

**Lemma 6.36.** The theorem is true if $V = V'$.

**Proof.** Let $\phi : U \to V$ be an orientation preserving parameterization. If $\omega_1 \sim \omega_2$, then

$$\int \phi^* \omega_1 = \int \phi^* \omega_2, \tag{6.102}$$

which is the same as saying that

$$\phi^* \omega_1 - \phi^* \omega_2 \in d\Omega^{n-1}_c(U), \tag{6.103}$$

which is the same as saying that

$$\omega_1 - \omega_2 \in d\Omega^{n-1}_c(V). \tag{6.104}$$

\hfill \Box

Step 5: In general, by the Connectivity Lemma, there exists sets $W_i$, $i = 0, \ldots, N+1$, such that each $W_i$ is diffeomorphic to an open set in $\mathbb{R}^n$. We can choose $W_0 = V$ and $W_{N+1} = V'$ and $W_i \cap W_{i+1} \neq \emptyset$ (where $\emptyset$ here is the empty set).

We can choose $\mu_i \in \Omega^n_c(W_i \cap W_{i+1})$ such that

$$c = \int_V \omega_1 = \int \mu_0 = \cdots = \int \mu_{N+1} = \int_{V'} \omega_2. \tag{6.105}$$
So,
\[
\omega_1 \sim \mu_0 \sim \cdots \sim \mu_N \sim \omega_2. \tag{6.106}
\]
We know that \(\mu_0 - \omega_1 \in d\Omega^{n-1}_c\) and \(\omega_2 - \mu_{N+1} \in d\Omega^{n-1}_C\). Also, each difference \(\omega_i - \omega_{i+1} \in d\Omega^{n-1}_c\). Therefore, \(\omega_1 - \omega_2 \in d\Omega^{n-1}_c\).

## 6.8 Degree on Manifolds

Suppose that \(X_1, X_2\) are oriented \(n\)-dimensional manifolds, and let \(f : X_1 \to X_2\) be a proper map (that is, for every compact set \(A \subseteq X\), the set pre-image \(f^{-1}(A)\) is compact). It follows that if \(\omega \in \Omega^k_c(X_2)\), then \(f^*\omega \in \Omega^k_c(X_1)\).

**Theorem 6.37.** If \(X_1, X_2\) are connected and \(f : X_1 \to X_2'\) is a proper \(C^\infty\) map, then there exists a topological invariant of \(f\) (called the degree of \(f\)) written \(\deg(f)\) such that for every \(\omega \in \Omega^k_c(X_2)\),

\[
\int_{X_1} f^*\omega = \deg(f) \int_{X_2} \omega. \tag{6.107}
\]

**Proof.** The proof is pretty much verbatim of the proof in Euclidean space.

Let us look at a special case. Let \(\phi_1 : U \to V\) be an oriented parameterization, and let \(V_1\) be open in \(X_1\). Let \(f : X_1 \to X_2\) be an oriented diffeomorphism. Define \(\phi_2 = f \circ \phi_1\), which is of the form \(\phi_2 : U \to V_2\), where \(V_2 = f(V_1)\). Notice that \(\phi_2\) is an oriented parameterization of \(V_2\).

Take \(\omega \in \Omega^k_c(V_2)\) and compute the integral

\[
\int_{V_1} f^*\omega = \int_U \phi_1^* f^*\omega \\
= \int_U (f \circ \phi_1)^* \omega \\
= \int_U \phi_2^* \omega. \tag{6.108}
\]

The \(n\)-form \(\omega\) is compactly supported on \(V_2\), so

\[
\int_{V_1} f^*\omega = \int_U \phi_2^* \omega \\
= \int_{X_2} \omega. \tag{6.109}
\]

On the other hand,

\[
\int_{X_1} f^*\omega = \int_{V_1} f^*\omega. \tag{6.110}
\]
Combining these results,
\[
\int_{X_1} f^* \omega = \int_{X_2} \omega. \tag{6.111}
\]
Therefore,
\[
\deg(f) = 1. \tag{6.112}
\]

So, we have proved the following theorem, which is the Change of Variables theorem for manifolds:

**Theorem 6.38.** Let $X_1, X_2$ be connected oriented $n$-dimensional manifolds, and let $f : X_1 \to X_2$ be an orientation preserving diffeomorphism. Then, for all $\omega \in \Omega^n_c(X_2)$,
\[
\int_{X_1} f^* \omega = \int_{X_2} \omega. \tag{6.113}
\]