Lecture 36

The first problem on today’s homework will be to prove the inverse function theorem for manifolds. Here we state the theorem and provide a sketch of the proof.

Let $X, Y$ be $n$-dimensional manifolds, and let $f : X \rightarrow Y$ be a $\mathcal{C}^\infty$ map with $f(p) = p_1$.

**Theorem 6.39.** If $d\!f_p : T_pX \rightarrow T_{p_1}Y$ is bijective, then $f$ maps a neighborhood $V$ of $p$ diffeomorphically onto a neighborhood $V_1$ of $p_1$.

**Sketch of proof:** Let $\phi : U \rightarrow V$ be a parameterization of $X$ at $p$, with $\phi(q) = p$. Similarly, let $\phi_1 : U_1 \rightarrow V_1$ be a parameterization of $Y$ at $p_1$, with $\phi_1(q_1) = p_1$.

Show that we can assume that $f : V \rightarrow V_1$ (Hint: if not, replace $V$ by $V \cap f^{-1}(V_1)$).

Show that we have a diagram

\[
\begin{array}{ccc}
V & \xrightarrow{f} & V_1 \\
\phi \uparrow & & \phi_1 \uparrow \\
U & \xrightarrow{g} & U_1,
\end{array}
\]  

which defines $g$,

\[
g = \phi_1^{-1} \circ f \circ \phi,
\]  

\[
g(q) = q_1.
\]

So,

\[
(dg)_q = (d\phi_1)_q^{-1} \circ d\!f_p \circ (d\phi)_q.
\]

Note that all three of the linear maps on the r.h.s. are bijective, so $(dg)_q$ is a bijection. Use the Inverse Function Theorem for open sets in $\mathbb{R}^n$. 

This ends our explanation of the first homework problem.

Last time we showed the following. Let $X, Y$ be $n$-dimensional manifolds, and let $f : X \rightarrow Y$ be a proper $\mathcal{C}^\infty$ map. We can define a topological invariant $\text{deg}(f)$ such that for every $\omega \in \Omega^n_c(Y)$,

\[
\int_X f^*\omega = \text{deg}(f) \int_Y \omega.
\]

There is a recipe for calculating the degree, which we state in the following theorem. We lead into the theorem with the following lemma.

First, remember that we defined the set $C_f$ of critical points of $f$ by

\[
p \in C_f \iff d\!f_p : T_pX \rightarrow T_qY \text{ is not surjective},
\]

where $q = f(p)$.  

1
Lemma 6.40. Suppose that \( q \in Y - f(C_f) \). Then \( f^{-1}(q) \) is a finite set.

Proof. Take \( p \in f^{-1}(q) \). Since \( p \notin C_f \), the map \( df_p \) is bijective. The Inverse Function Theorem tells us that \( f \) maps a neighborhood \( U_p \) of \( p \) diffeomorphically onto an open neighborhood of \( q \). So, \( U_p \cap f^{-1}(q) = p \).

Next, note that \( \{ U_p : p \in f^{-1}(q) \} \) is an open covering of \( f^{-1}(q) \). Since \( f \) is proper, \( f^{-1}(q) \) is compact, so there exists a finite subcover \( U_{p_1}, \ldots, U_{p_N} \). Therefore, \( f^{-1}(q) = \{ p_1, \ldots, p_N \} \).

The following theorem gives a recipe for computing the degree.

Theorem 6.41. \[
\deg(f) = \sum_{i=1}^{N} \sigma_{p_i},
\]

where
\[
\sigma_{p_i} = \begin{cases} +1 & \text{if } df_{p_i} : T_{p_i}X \to T_qY \text{ is orientation preserving,} \\ -1 & \text{if } df_{p_i} : T_{p_i}X \to T_qY \text{ is orientation reversing,} \end{cases}
\]

Proof. The proof is basically the same as the proof in Euclidean space.

We say that \( q \in Y \) is a regular value of \( f \) if \( q \notin f(C_f) \). Do regular values exist? We showed that in the Euclidean case, the set of non-regular values is of measure zero (Sard’s Theorem). The following theorem is the analogous theorem for manifolds.

Theorem 6.42. If \( q_0 \in Y \) and \( W \) is a neighborhood of \( q_0 \) in \( Y \), then \( W - f(C_f) \) is non-empty. That is, every neighborhood of \( q_0 \) contains a regular value (this is known as the Volume Theorem).

Proof. We reduce to Sard’s Theorem.

The set \( f^{-1}(q_0) \) is a compact set, so we can cover \( f^{-1}(q_0) \) by open sets \( V_i \subset X \), \( i = 1, \ldots, N \), such that each \( V_i \) is diffeomorphic to an open set in \( \mathbb{R}^n \).

Let \( W \) be a neighborhood of \( q_0 \) in \( Y \). We can assume the following:

1. \( W \) is diffeomorphic to an open set in \( \mathbb{R}^n \),
2. \( f^{-1}(W) \subset \bigcup V_i \) (which is Theorem 4.3 in the Supp. Notes),
3. \( f(V_i) \subset W \) (for, if not, we can replace \( V_i \) with \( V_i \cap f^{-1}(W) \)).

Let \( U \) and the sets \( U_i \), \( i = 1, \ldots, N \), be open sets in \( \mathbb{R}^n \). Let \( \phi : U \to W \) and the maps \( \phi_i : U_i \to V_i \) be diffeomorphisms. We have the following diagram:
\[
\begin{array}{ccc}
V_i & \xrightarrow{f} & W \\
\phi_i \uparrow & & \phi \uparrow \\
U_i & \xrightarrow{g_i} & U,
\end{array}
\]
which define the maps $g_i$,
\[ g_i = \phi^{-1} \circ f \circ \phi. \] (6.123)
By the chain rule, $x \in C_{g_i} \implies \phi_i(x) \in C_f$, so
\[ \phi_i(C_{g_i}) = C_f \cap V_i. \] (6.124)
So,
\[ \phi(g_i(C_{g_i})) = f(C_f \cap V_i). \] (6.125)
Then,
\[ f(C_f) \cap W = \bigcup_i \phi(g_i(C_{g_i})). \] (6.126)
Sard’s Theorem tells us that $g_i(C_{g_i})$ is a set of measure zero in $U$, so
\[ U - \bigcup g_i(C_{g_i}) \text{ is non-empty}, \] (6.127)
\[ W - f(C_f) \text{ is also non-empty}. \] (6.128)
In fact, this set is not only non-empty, but is a very, very “full” set.

Let $f_0, f_1 : X \to Y$ be proper $C^\infty$ maps. Suppose there exists a proper $C^\infty$ map $F : X \times [0,1] \to Y$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. Then
\[ \deg(f_0) = \deg(f_1). \] (6.129)
In other words, the degree is a homotopy. The proof of this is essential the same as before.

6.9 Hopf Theorem

The Hopf Theorem is a nice application of the homotopy invariance of the degree.

Define the $n$-sphere
\[ S^n = \{ v \in \mathbb{R}^{n+1} : ||v|| = 1 \}. \] (6.130)

**Hopf Theorem.** Let $n$ be even. Let $f : S^n \to \mathbb{R}^{n+1}$ be a $C^\infty$ map. Then, for some $v \in S^n$,
\[ f(v) = \lambda v, \] (6.131)
for some scalar $\lambda \in \mathbb{R}$.

**Proof.** We prove the contrapositive. Assume that no such $v$ exists, and take $w = f(v)$. Consider $w - \langle v, w \rangle v \equiv w - w_1$. It follows that $w - w_1 \neq 0$.

Define a new map $\tilde{f} : S^n \to S^n$ by
\[ \tilde{f}(v) = \frac{f(v) - \langle v, f(x) \rangle}{||f(v) - \langle v, f(x) \rangle||}. \] (6.132)
Note that \((w - w_1) \perp v\), so \(\tilde{f}(v) \perp v\).

Define a family of functions

\[
\begin{align*}
  f_t &: S^n \to S^n, \\
  f_t(v) &= (\cos t)v + (\sin t)\tilde{w},
\end{align*}
\]

where \(\tilde{w} = \tilde{f}(v)\) has the properties \(||\tilde{w}|| = 1\) and \(\tilde{w} \perp v\).

We compute the degree of \(f_t\). When \(t = 0\), \(f_t = \text{id}\), so

\[
\deg(f_t) = \deg(f_0) = 1.
\]

When \(t = \pi\), \(f_t(v) = -v\). But, if \(n\) is even, a map from \(S^n \to S^n\) mapping \(v \to (-v)\) has degree \(-1\). We have arrived at a contradiction. \(\square\)