SOLUTIONS TO PROBLEM SET 4 FOR 18.102, SPRING 2009
WAS DUE 11AM TUESDAY 10 MAR.

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Just to compensate for last week, I will make this problem set too short and easy!

1. Problem 4.1

Let $H$ be a normed space in which the norm satisfies the parallelogram law:

(1) $\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2) \forall u, v \in H.$

Show that the norm comes from a positive definite sesquilinear (i.e. Hermitian) inner product. Big Hint:- Try

(2) $(u, v) = \frac{1}{4} (\|u + v\|^2 - \|u - v\|^2 + i\|u + iv\|^2 - i\|u - iv\|^2)!$

Solution: Setting $u = v$, even without the parallelogram law,

(3) $(u, u) = \frac{1}{4} (\|2u\|^2 + i\|(1 + i)u\|^2 - i\|(1 - i)u\|^2) = \|u\|^2.$

So the point is that the parallelogram law shows that $(u, v)$ is indeed an Hermitian inner product. Taking complex conjugates and using properties of the norm, $\|u + iv\| = \|v - iu\|$ etc

(4) $(u, v) = \frac{1}{4} (\|v + u\|^2 - \|v - u\|^2 - i\|v - iv\|^2 + i\|v + iv\|^2) = (v, u).$

Thus we only need check the linearity in the first variable. This is a little tricky! First compute away. Directly from the identity $(u, -v) = -(u, v)$ so $(-u, v) = -(u, v)$ using (4). Now,

(5) $(2u, v) = \frac{1}{4} (\|u + (u + v)\|^2 - \|u + (u - v)\|^2 + i\|u + (u + iv)\|^2 - i\|u + (u - iv)\|^2) = \frac{1}{4} (\|u + v\|^2 + \|u + \|u\|^2 - \|u - v\|^2 - \|u\|^2 + i\|u + iv\|^2 + i\|u\|^2 - i\|u - iv\|^2 - i\|v\|^2) = \frac{1}{4} (\|u - (u, v).$

Using this and (4), for any $u$, $w$ and $v$,

(6) $(u + w, v) = \frac{1}{2} (u + w, 2v) = \frac{1}{2} (\|u + v\|^2 + (u + v)\|^2 - \|(u - v) + (u' - v)\|^2 + i\|(u + iv) + (u - iv)\|^2) - i\|(u - iv)\|^2 - i\|(u - iv)\|^2 - i\|v\|^2) = \frac{1}{4} (\|u + (u + v)\|^2 - \|u + (u - v)\|^2 + i\|u + (u + iv)\|^2 + i\|u\|^2 - i\|u - iv\|^2 - i\|v\|^2) - \frac{1}{4} (\|u - (u, v).$

Using the second identity to iterate the first it follows that $(ku, v) = k(u, v)$ for any $u$ and $v$ and any positive integer $k$. Then setting $ru' = u$ for any other positive integer and $r = k/n$, it follows that

(7) $(ru, v) = (ku', v) = k(u', v) = r(n(u', v) = r(u, v)$
where the identity is reversed. Thus it follows that \((ru, v) = r(u, v)\) for any rational \(r\). Now, from the definition both sides are continuous in the first element, with respect to the norm, so we can pass to the limit as \(r \to x\) in \(\mathbb{R}\). Also directly from the definition,
\[
(8) \quad (iu, v) = \frac{1}{4} (\|iu + v\|^2 - \|iu - v\|^2 + i\|iu + iv\|^2 - i\|iu - iv\|^2) = i(u, v)
\]
so now full linearity in the first variable follows and that is all we need.

2. **Problem 4.2**

Let \(H\) be a finite dimensional (pre)Hilbert space. So, by definition \(H\) has a basis \(\{v_i\}_{i=1}^n\), meaning that any element of \(H\) can be written
\[
(1) \quad v = \sum_i c_i v_i
\]
and there is no dependence relation between the \(v_i\)'s – the presentation of \(v = 0\) in the form (1) is unique. Show that \(H\) has an orthonormal basis, \(\{e_i\}_{i=1}^n\) satisfying \((e_i, e_j) = \delta_{ij} (= 1\) if \(i = j\) and 0 otherwise). Check that for the orthonormal basis the coefficients in (1) are \(c_i = (v, e_i)\) and that the map
\[
(2) \quad T : H \ni v \longmapsto ((v, e_i)) \in \mathbb{C}^n
\]
is a linear isomorphism with the properties
\[
(3) \quad (u, v) = \sum_i (Tu)_i \bar{(Tv)}_i, \quad \|u\|_H = \|Tu\|_{\mathbb{C}^n} \forall u, v \in H.
\]

Why is a finite dimensional preHilbert space a Hilbert space?

Solution: Since \(H\) is assumed to be finite dimensional, it has a basis \(v_i, i = 1, \ldots, n\). This basis can be replaced by an orthonormal basis in \(n\) steps. First replace \(v_1\) by \(e_1 = v_1/\|v_1\|\) where \(\|v_1\| \neq 0\) by the linear independance of the basis. Then replace \(v_2\) by
\[
(4) \quad e_2 = w_2/\|w_2\|, \quad w_2 = v_2 - \langle v_2, e_1 \rangle e_1.
\]
Here \(w_2 \perp e_1\) as follows by taking inner products; \(w_2\) cannot vanish since \(v_2\) and \(e_1\) must be linearly independent. Proceeding by finite induction we may assume that we have replaced \(v_1, v_2, \ldots, v_k, k < n\), by \(e_1, e_2, \ldots, e_k\) which are orthonormal and span the same subspace as the \(v_i\)'s \(i = 1, \ldots, k\). Then replace \(v_{k+1}\) by
\[
(5) \quad e_{k+1} = w_{k+1}/\|w_{k+1}\|, \quad w_{k+1} = v_{k+1} - \sum_{i=1}^k \langle v_{k+1}, e_i \rangle e_i.
\]
By taking inner products, \(w_{k+1} \perp e_i, i = 1, \ldots, k\) and \(w_{k+1} \neq 0\) by the linear independence of the \(v_i\)'s. Thus the orthonormal set has been increased by one element preserving the same properties and hence the basis can be orthonormalized.

Now, for each \(u \in H\) set
\[
(6) \quad c_i = \langle u, e_i \rangle.
\]
It follows that \(U = u - \sum_{i=1}^n c_i e_i\) is orthogonal to all the \(e_i\) since
\[
(7) \quad \langle u, e_j \rangle = \langle u, e_j \rangle - \sum_i c_i \langle e_i, e_j \rangle = \langle u, e_j \rangle - c_j = 0.
\]
This implies that \( U = 0 \) since writing \( U = \sum d_i e_i \) it follows that \( d_i = \langle U, e_i \rangle = 0 \).

Now, consider the map (2). We have just shown that this map is injective, since \( Tu = 0 \) implies \( c_i = 0 \) for all \( i \) and hence \( u = 0 \). It is linear since the \( c_i \) depend linearly on \( u \) by the linearity of the inner product in the first variable. Moreover it is surjective, since for any \( c_i \in \mathbb{C} \), \( u = \sum c_i e_i \) reproduces the \( c_i \) through (6). Thus \( T \) is a linear isomorphism and the first identity in (3) follows by direct computation:

\[
\sum_{i=1}^{n} (Tu)_i (Tv)_i = \sum_i \langle u, e_i \rangle \\
= \langle u, \sum_i \langle v, e_i \rangle e_i \rangle \\
= \langle u, v \rangle.
\]

(8)

Setting \( u = v \) shows that \( \| Tu \|_{\mathbb{C}^n} = \| u \|_{H} \).

Now, we know that \( \mathbb{C}^n \) is complete with its standard norm. Since \( T \) is an isomorphism, it carries Cauchy sequences in \( H \) to Cauchy sequences in \( \mathbb{C}^n \) and \( T^{-1} \) carries convergent sequences in \( \mathbb{C}^n \) to convergent sequences in \( H \), so every Cauchy sequence in \( H \) is convergent. Thus \( H \) is complete.

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