18.102 Introduction to Functional Analysis
Spring 2009

SOLUTIONS TO PROBLEM SET 4 FOR 18.102, SPRING 2009 WAS DUE 11AM TUESDAY 10 MAR.

RICHARD MELROSE

Just to compensate for last week, I will make this problem set too short and easy!

1. Problem 4.1

Let H be a normed space in which the norm satisfies the parallelogram law:

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2) \ \forall\ u, v \in H. \tag{1}$$

Show that the norm comes from a positive definite sesquilinear (i.e. Hermitian) inner product. Big Hint:- Try

$$(u,v) = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2\right)! \tag{2}$$

Solution: Setting $u = v$, even without the parallelogram law,

$$(u,u) = \frac{1}{4}\left\|2u\|^2 + i\|(1+i)u\|^2 - i\|(1-i)u\|^2\right) = \|u\|^2. \tag{3}$$

So the point is that the parallelogram law shows that (u,v) is indeed an Hermitian inner product. Taking complex conjugates and using properties of the norm, $\|u + iv\| = \|v - iu\|$ etc

$$\overline{(u,v)} = \frac{1}{4}\left(\|v+u\|^2 - \|v-u\|^2 - i\|v-iu\|^2 + i\|v+iu\|^2\right) = (v,u). \tag{4}$$

Thus we only need check the linearity in the first variable. This *is* a little tricky! First compute away. Directly from the identity $(u,-v) = -(u,v)$ so $(-u,v) = -(u,v)$ using (4). Now,

(5)

$$\begin{aligned}(2u,v) &= \frac{1}{4}\left(\|u+(u+v)\|^2 - \|u+(u-v)\|^2 + i\|u+(u+iv)\|^2 - i\|u+(u-iv)\|^2\right)\\ &= 2\frac{1}{4}\left(\|u+v\|^2 + \|u\|^2 - \|u-v\|^2 - \|u\|^2 + i\|(u+iv)\|^2 + i\|u\|^2 - i\|u-iv\|^2 - i\|u\|^2\right) - \frac{1}{4}\left(\|u-(u\right.\\ &= 2(u,v).\end{aligned}$$

Using this and (4), for any u, u' and v,

(6)

$$(u+u',v) = \frac{1}{2}(u+u',2v) = \frac{1}{2}\frac{1}{4}\left(\|(u+v)+(u'+v)\|^2 - \|(u-v)+(u'-v)\|^2 + i\|(u+iv)+(u-iv)\|^2 - i\|(\right.$$

Using the second identity to iterate the first it follows that $(ku,v) = k(u,v)$ for any u and v and any positive integer k. Then setting $nu' = u$ for any other positive integer and $r = k/n$, it follows that

$$(ru,v) = (ku',v) = k(u',v) = rn(u',v) = r(u,v) \tag{7}$$

where the identity is reversed. Thus it follows that $(ru, v) = r(u, v)$ for any rational r. Now, from the definition both sides are continuous in the first element, with respect to the norm, so we can pass to the limit as $r \to x$ in $\mathbb{R}$. Also directly from the definition,

$$(8) \qquad (iu, v) = \frac{1}{4}\left(\|iu+v\|^2 - \|iu-v\|^2 + i\|iu+iv\|^2 - i\|iu-iv\|^2\right) = i(u, v)$$

so now full linearity in the first variable follows and that is all we need.

2. Problem 4.2

Let H be a finite dimensional (pre)Hilbert space. So, by definition H has a basis $\{v_i\}_{i=1}^n$, meaning that any element of H can be written

$$(1) \qquad v = \sum_i c_i v_i$$

and there is no dependence relation between the v_i's – the presentation of $v = 0$ in the form (1) is unique. Show that H has an orthonormal basis, $\{e_i\}_{i=1}^n$ satisfying $(e_i, e_j) = \delta_{ij}$ ($= 1$ if $i = j$ and 0 otherwise). Check that for the orthonormal basis the coefficients in (1) are $c_i = (v, e_i)$ and that the map

$$(2) \qquad T : H \ni v \longmapsto ((v, e_i)) \in \mathbb{C}^n$$

is a linear isomorphism with the properties

$$(3) \qquad (u, v) = \sum_i (Tu)_i \overline{(Tv)_i}, \ \|u\|_H = \|Tu\|_{\mathbb{C}^n} \ \forall \ u, v \in H.$$

Why is a finite dimensional preHilbert space a Hilbert space?

Solution: Since H is assumed to be finite dimensional, it has a basis v_i, $i = 1, \dots, n$. This basis can be replaced by an orthonormal basis in n steps. First replace v_1 by $e_1 = v_1/\|v_1\|$ where $\|v_1\| \neq 0$ by the linear indepedence of the basis. Then replace v_2 by

$$(4) \qquad e_2 = w_2/\|w_2\|, \ w_2 = v_2 - \langle v_2, e_1\rangle e_1.$$

Here $w_2 \perp e_1$ as follows by taking inner products; w_2 cannot vanish since v_2 and e_1 must be linearly independent. Proceeding by finite induction we may assume that we have replaced v_1, v_2, $\dots$, v_k, $k < n$, by e_1, e_2, $\dots$, e_k which are orthonormal and span the same subspace as the v_i's $i = 1, \dots, k$. Then replace v_{k+1} by

$$(5) \qquad e_{k+1} = w_{k+1}/\|w_{k+1}\|, \ w_{k+1} = v_{k+1} - \sum_{i=1}^k \langle v_{k+1}, e_i\rangle e_i.$$

By taking inner products, $w_{k+1} \perp e_i$, $i = 1, \dots, k$ and $w_{k+1} \neq 0$ by the linear independence of the v_i's. Thus the orthonormal set has been increased by one element preserving the same properties and hence the basis can be orthonormalized.

Now, for each $u \in H$ set

$$(6) \qquad c_i = \langle u, e_i\rangle.$$

It follows that $U = u - \sum\limits_{i=1}^n c_i e_i$ is orthogonal to all the e_i since

$$(7) \qquad \langle u, e_j\rangle = \langle u, e_j\rangle - \sum_i c_i \langle e_i, e_j\rangle = \langle u.e_j\rangle - c_j = 0.$$

This implies that $U = 0$ since writing $U = \sum_i d_i e_i$ it follows that $d_i = \langle U, e_i \rangle = 0$.

Now, consider the map (2). We have just shown that this map is injective, since $Tu = 0$ implies $c_i = 0$ for all i and hence $u = 0$. It is linear since the c_i depend linearly on u by the linearity of the inner product in the first variable. Moreover it is surjective, since for any $c_i \in \mathbb{C}$, $u = \sum_i c_i e_i$ reproduces the c_i through (6). Thus T is a linear isomorphism and the first identity in (3) follows by direct computation:-

$$\begin{aligned} \sum_{i=1}^{n} (Tu)_i \overline{(Tv)_i} &= \sum_i \langle u, e_i \rangle \\ &= \langle u, \sum_i \langle v, e_i \rangle e_i \rangle \\ &= \langle u, v \rangle. \end{aligned} \tag{8}$$

Setting $u = v$ shows that $\|Tu\|_{\mathbb{C}^n} = \|u\|_H$.

Now, we know that $\mathbb{C}^n$ is complete with its standard norm. Since T is an isomorphism, it carries Cauchy sequences in H to Cauchy sequences in $\mathbb{C}^n$ and T^{-1} carries convergent sequences in $\mathbb{C}^n$ to convergent sequences in H, so every Cauchy sequence in H is convergent. Thus H is complete.

Department of Mathematics, Massachusetts Institute of Technology