You should be thinking about using Lebesgue's dominated convergence at several points below.

**Problem 5.1**

Let $f : \mathbb{R} \to \mathbb{C}$ be an element of $L^1(\mathbb{R})$. Define

$$f_L(x) = \begin{cases} f(x) & x \in [-L, L] \\ 0 & \text{otherwise.} \end{cases}$$

Show that $f_L \in L^1(\mathbb{R})$ and that $\int |f_L - f| \to 0$ as $L \to \infty$.

Solution. If $\chi_L$ is the characteristic function of $[-N, N]$ then $f_L = f\chi_L$. If $f_n$ is an absolutely summable series of step functions converging a.e. to $f$ then $f_n\chi_L$ is absolutely summable, since $\int |f_n\chi_L| \leq \int |f_n|$ and converges a.e. to $f_L$, so $f_L \in L^1(\mathbb{R})$. Certainly $|f_L(x) - f(x)| \to 0$ for each $x$ as $L \to \infty$ and $|f_L(x) - f(x)| \leq |f(x)| + |f(x)| \leq 2|f(x)|$ so by Lebesgue's dominated convergence, $\int |f - f_L| \to 0$.

**Problem 5.2**

Consider a real-valued function $f : \mathbb{R} \to \mathbb{R}$ which is locally integrable in the sense that

$$g_L(x) = \begin{cases} f(x) & x \in [-L, L] \\ 0 & x \in \mathbb{R} \setminus [-L, L] \end{cases}$$

is Lebesgue integrable of each $L \in \mathbb{N}$.

1. Show that for each fixed $L$ the function

$$g_L^{(N)}(x) = \begin{cases} g_L(x) & \text{if } g_L(x) \in [-N, N] \\ N & \text{if } g_L(x) > N \\ -N & \text{if } g_L(x) < -N \end{cases}$$

is Lebesgue integrable.

2. Show that $\int |g_L^{(N)} - g_L| \to 0$ as $N \to \infty$.

3. Show that there is a sequence, $h_n$, of step functions such that

$$h_n(x) \to f(x) \text{ a.e. in } \mathbb{R}.$$ 

4. Defining

$$h_n^{(N)}_{n,L} = \begin{cases} 0 & x \notin [-L, L] \\ h_n(x) & \text{if } h_n(x) \in [-N, N], x \in [-L, L] \\ N & \text{if } h_n(x) > N, x \in [-L, L] \\ -N & \text{if } h_n(x) < -N, x \in [-L, L] \end{cases},$$

was due 11am Tuesday 17 Mar.

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Show that \( \int |h_{n,L}^{(N)} - g_L^{(N)}| \to 0 \) as \( n \to \infty \).

Solution:

(1) By definition \( g_L^{(N)} = \max(-N \chi_L, \min(N \chi_L, g_L)) \) where \( \chi_L \) is the characteristic function of \([-L, L] \), thus it is in \( L^1(\mathbb{R}) \).

(2) Clearly \( g_L^{(N)}(x) \to g_L(x) \) for every \( x \) and \( |g_L^{(N)}(x)| \leq |g_L(x)| \) so by Dominated Convergence, \( g_L^{(N)} \to g_L \) in \( L^1 \), i.e. \( \int |g_L^{(N)} - g_L| \to 0 \) as \( N \to \infty \) since the sequence converges to 0 pointwise and is bounded by \( 2|g(x)| \).

(3) Let \( S_{L,n} \) be a sequence of step functions converging a.e. to \( g_L \) – for example the sequence of partial sums of an absolutely summable series of step functions converging to \( g_L \) which exists by the assumed integrability. Then replacing \( S_{L,n} \) by \( S_{L,n \chi_L} \) we can assume that the elements all vanish outside \([-N, N]\) but still have convergence a.e. to \( g_L \). Now take the sequence

\[
(5.6) \quad h_n(x) = \begin{cases} S_{k,n-k} & \text{on } [k,-k] \setminus [(k-1), -(k-1)], \ 1 \leq k \leq n, \\ 0 & \text{on } \mathbb{R} \setminus [-n,n]. \end{cases}
\]

This is certainly a sequence of step functions – since it is a finite sum of step functions for each \( n \) – and on \([-L, L] \setminus (L-1), (L-1)\] for large integral \( L \) is just \( S_{L,n-L} \to g_L \). Thus \( h_n(x) \to f(x) \) outside a countable union of sets of measure zero, so also almost everywhere.

(4) This is repetition of the first problem, \( h_{n,L}^{(N)}(x) \to g_L^{(N)} \) almost everywhere and \( |h_{n,L}^{(N)}| \leq N \chi_L \) so \( g_L^{(N)} \in \mathcal{L}^1(\mathbb{R}) \) and \( \int |h_{n,L}^{(N)} - g_L^{(N)}| \to 0 \) as \( n \to \infty \).

**Problem 5.3**

Show that \( \mathcal{L}^2(\mathbb{R}) \) is a Hilbert space – since it is rather central to the course I wanted you to go through the details carefully!

First working with real functions, define \( \mathcal{L}^2(\mathbb{R}) \) as the set of functions \( f : \mathbb{R} \to \mathbb{R} \) which are locally integrable and such that \( |f|^2 \) is integrable.

(1) For such \( f \) choose \( h_n \) and define \( g_L \), \( g_L^{(N)} \) and \( h_{n,L}^{(N)} \) by \((5.2), (5.3) \) and \( (5.5) \).

(2) Show using the sequence \( h_{n,L}^{(N)} \) for fixed \( N \) and \( \mathbb{L} \) that \( g_L^{(N)} \) and \( (g_L^{(N)})^2 \) are in \( \mathcal{L}^1(\mathbb{R}) \) and that \( \int (|h_{n,L}^{(N)}|^2 - (g_L^{(N)})^2) \to 0 \) as \( n \to \infty \).

(3) Show that \( (g_L)^2 \in \mathcal{L}^1(\mathbb{R}) \) and that \( \int (|g_L^{(N)})^2 - (g_L)^2) \to 0 \) as \( N \to \infty \).

(4) Show that \( \int (|g_L)^2 - f^2) \to 0 \) as \( \mathbb{L} \to \infty \).

(5) Show that \( f, g \in \mathcal{L}^2(\mathbb{R}) \) then \( fg \in \mathcal{L}^1(\mathbb{R}) \) and that

\[
(5.7) \quad |\int fg| \leq \int |fg| \leq \|f\|_{L^2}\|g\|_{L^2}, \ |f|^2_{L^2} = \int |f|^2.
\]

(6) Use these constructions to show that \( \mathcal{L}^2(\mathbb{R}) \) is a linear space.

(7) Conclude that the quotient space \( \mathcal{L}^2(\mathbb{R}) = \mathcal{L}^2(\mathbb{R})/\mathcal{N} \), where \( \mathcal{N} \) is the space of null functions, is a real Hilbert space.

(8) Extend the arguments to the case of complex-valued functions.

Solution:

(1) Done. I think it should have been \( h_{n,L}^{(N)} \).
(2) We already checked that $g^{(N)}_L \in L^1(\mathbb{R})$ and the same argument applies to $(g^{(N)}_L)^2$ namely $(h^{(N)}_{L,N})^2 \to g^{(N)}_L$ almost everywhere and both are bounded by $N^2 \chi_L$ so by dominated convergence 
\[ (h^{(N)}_{L,N})^2 \to g^{(N)}_L \text{ a.e.} \implies g^{(N)}_L \in L^1(\mathbb{R}) \text{ and} \]
\[
|h^{(N)}_{L,N}(x)|^2 - g^{(N)}_L(x)|^2| \to 0 \text{ a.e.,}
\]
\[
|h^{(N)}_{L,N}(x)|^2 - g^{(N)}_L(x)|^2| \leq 2N^2\chi_L \implies \int |h^{(N)}_{L,N}(x)|^2 - g^{(N)}_L(x)|^2| \to 0.
\]

(3) Now, as $N \to \infty$, $(g^{(N)}_L)^2 \to (g_L)^2$ a.e. and $(g^{(N)}_L)^2 \to (g_L)^2 \leq f^2$ so by dominated convergence, $(g_L)^2 \in L^1(\mathbb{R})$ and $\int |(g^{(N)}_L)^2 - (g_L)^2| \to 0$ as $N \to \infty$.

(4) The same argument of dominated convergence shows now that $g^{(N)}_L \to f^2$ and $\int |g^{(N)}_L|^2 - f^2| \to 0$ using the bound by $f^2 \in L^1(\mathbb{R})$.

(5) What this is all for is to show that $f, g \in L^1(\mathbb{R})$ if $f, F = g \in L^2(\mathbb{R})$ (for easier notation). Approximate each of them by sequences of step functions as above, $h^{(N)}_f$ for $f$ and $h^{(N)}_g$ for $g$. Then the product sequence is in $L^1$ - being a sequence of step functions - and
\[
h^{(N)}_f(x)H^{(N)}_g(x) \to g^{(N)}_L(x)G^{(N)}_L(x)
\]
almost everywhere and with absolute value bounded by $N^2\chi_L$. Thus by dominated convergence $g^{(N)}_L G^{(N)}_L \in L^1(\mathbb{R})$. Now, let $N \to \infty$; this sequence converges almost everywhere to $g_L(x)G_L(x)$ and we have the bound
\[ |g^{(N)}_L(x)G^{(N)}_L(x)| \leq |f(x)F(x)|\frac{1}{2}(f^2 + F^2)
\]
so as always by dominated convergence, the limit $g_L G_L \in L^1$. Finally, letting $L \to \infty$ the same argument shows that $fF \in L^1(\mathbb{R})$. Moreover, $|fF| \in L^1(\mathbb{R})$ and
\[
|\int fF| \leq \int |fF| \leq \|f\|_{L^2}\|F\|_{L^2}
\]
where the last inequality follows from Cauchy's inequality - if you wish, first for the approximating sequences and then taking limits.

(6) So if $f, g \in L^2(\mathbb{R})$ are real-value, $f + g$ is certainly locally integrable and
\[ (f + g)^2 = f^2 + 2fg + g^2 \in L^1(\mathbb{R})
\]
by the discussion above. For constants $f \in L^2(\mathbb{R})$ implies $cf \in L^2(\mathbb{R})$ is directly true.

(7) The argument is the same as for $L^1$ versus $L^1$. Namely $\int f^2 = 0$ implies that $f^2 = 0$ almost everywhere which is equivalent to $f = 0$ a.e. Then the norm is the same for all $f + h$ where $h$ is a null function since $fL$ and $h^2$ are null so $(f + h)^2 = f^2 + 2fh + h^2$. The same is true for the inner product so it follows that the quotient by null functions
\[ L^2(\mathbb{R}) = L^2(\mathbb{R})/N
\]
is a preHilbert space.

However, it remains to show completeness. Suppose $\{f_n\}$ is an absolutely summable series in $L^2(\mathbb{R})$ which means that $\sum \|f_n\|_{L^2} < \infty$. It
follows that the cut-off series $f_n \chi_L$ is absolutely summable in the $L^1$ sense since
\begin{equation}
\int |f_n \chi_L| \leq L^{\frac{1}{2}} \left( \int f_n^2 \right)^{\frac{1}{2}}
\end{equation}
by Cauchy’s inequality. Thus if we set $F_n = \sum_{k=1}^{n} f_k$ then $F_n(x) \chi_L$ converges almost everywhere for each $L$ so in fact
\begin{equation}
F_n(x) \rightarrow f(x) \text{ converges almost everywhere.}
\end{equation}
We want to show that $f \in L^2(\mathbb{R})$ where it follows already that $f$ is locally integrable by the completeness of $L^1$. Now consider the series
\begin{equation}
g_1 = F_1^2, \quad g_n = F_n^2 - F_{n-1}^2.
\end{equation}
The elements are in $L^1(\mathbb{R})$ and by Cauchy’s inequality for $n > 1$,
\begin{equation}
\int |g_n| = \int |F_n^2 - F_{n-1}^2| \leq \|F_n - F_{n-1}\|_{L^2} \|F_n + F_{n-1}\|_{L^2} \leq \|f_n\|_{L^2} \sum_k \|f_k\|_{L^2}
\end{equation}
where the triangle inequality has been used. Thus in fact the series $g_n$ is absolutely summable in $L^1$
\begin{equation}
\sum_n \int |g_n| \leq 2(\sum_n \|f_n\|_{L^2})^2.
\end{equation}
So indeed the sequence of partial sums, the $F_n^2$ converge to $f^2 \in L^1(\mathbb{R})$. Thus $f \in L^2(\mathbb{R})$ and moreover \begin{equation}
\int (F_n - f)^2 = \int F_n^2 + \int f^2 - 2 \int F_n f \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{equation}
Indeed the first term converges to $f^2$ and, by Cauchy’s inequality, the series of products $f_n f$ is absolutely summable in $L^1$ with limit $f^2$ so the third term converges to $-2 \int f^2$. Thus in fact $[F_n] \rightarrow [f]$ in $L^2(\mathbb{R})$ and we have proved completeness.

(8) For the complex case we need to check linearity, assuming $f$ is locally integrable and $|f|^2 \in L^1(\mathbb{R})$. The real part of $f$ is locally integrable and the approximation $F_L^{(N)}$ discussed above is square integrable with $(F_L^{(N)})^2 \leq |f|^2$ so by dominated convergence, letting first $N \rightarrow \infty$ and then $L \rightarrow \infty$ the real part is in $L^2(\mathbb{R})$. Now linearity and completeness follow from the real case.

**Problem 5.4**

Consider the sequence space
\begin{equation}
h^{2,1} = \left\{ c : \exists N \ni j \mapsto c_j \in \mathbb{C} ; \sum_j (1 + j^2) |c_j|^2 < \infty \right\}.
\end{equation}

(1) Show that
\begin{equation}
h^{2,1} \times h^{2,1} \ni (c, d) \mapsto (c, d) = \sum_j (1 + j^2) c_j d_j
\end{equation}
is an Hermitian inner form which turns $h^{2,1}$ into a Hilbert space.
(2) Denoting the norm on this space by $\| \cdot \|_{2,1}$ and the norm on $l^2$ by $\| \cdot \|_2$, show that

$$h^{2,1} \subset l^2, \quad \|c\|_2 \leq \|c\|_{2,1} \quad \forall \, c \in h^{2,1}.$$ 

**Solution:**

(1) The inner product is well defined since the series defining it converges absolutely by Cauchy's inequality:

$$\langle c, d \rangle = \sum_j (1 + j^2)^{\frac{1}{2}} c_j (1 + j^2)^{\frac{1}{2}} d_j,$$

and

$$\sum_j |(1 + j^2)^{\frac{1}{2}} c_j (1 + j^2)^{\frac{1}{2}} d_j| \leq \left( \sum_j (1 + j^2)^{\frac{1}{2}} |c_j|^2 \right) \left( \sum_j (1 + j^2)^{\frac{1}{2}} |d_j|^2 \right)^{\frac{1}{2}}.$$

It is sesquilinear and positive definite since

$$\|c\|_{2,1} = \left( \sum_j (1 + j^2)^{\frac{1}{2}} |c_j|^2 \right)^{\frac{1}{2}}$$

only vanishes if all $c_j$ vanish. Completeness follows as for $l^2$ — if $c^{(n)}$ is a Cauchy sequence then each component $c_j^{(n)}$ converges, since $(1 + j^2)^{\frac{1}{2}} c_j^{(n)}$ is Cauchy. The limits $c_j$ define an element of $h^{2,1}$ since the sequence is bounded and

$$\sum_{j=1}^N (1 + j^2)^{\frac{1}{2}} |c_j|^2 = \lim_{n \to \infty} \sum_{j=1}^N (1 + j^2)^{\frac{1}{2}} |c_j^{(n)}|^2 \leq A$$

where $A$ is a bound on the norms. Then from the Cauchy condition $c^{(n)} \to c$ in $h^{2,1}$ by passing to the limit as $m \to \infty$ in $\|c^{(n)} - c^{(m)}\|_{2,1} \leq \epsilon$.

(2) Clearly $h^{2,2} \subset l^2$ since for any finite $N$

$$\sum_{j=1}^N |c_j|^2 \sum_{j=1}^N (1 + j^2)^{\frac{1}{2}} |c_j|^2 \leq \|c\|_{2,1}^2$$

and we may pass to the limit as $N \to \infty$ to see that

$$\|c\|_2 \leq \|c\|_{2,1}.$$ 

**Problem 5.5**

In the separable case, prove Riesz Representation Theorem directly.

Choose an orthonormal basis $\{e_i\}$ of the separable Hilbert space $H$. Suppose $T : H \to \mathbb{C}$ is a bounded linear functional. Define a sequence

$$w_i = \overline{T(e_i)}, \quad i \in \mathbb{N}.$$ 

(1) Now, recall that $|Tu| \leq C||u||_H$ for some constant $C$. Show that for every finite $N$,

$$\sum_{j=1}^N |w_j|^2 \leq C^2.$$

(2) Conclude that $\{w_i\} \in l^2$ and that

$$w = \sum_i w_i e_i \in H.$$
(3) Show that

\[ T(u) = \langle u, w \rangle_H \text{ for all } u \in H \text{ and } \| T \| = \| w \|_H. \]

Solution:

(1) The finite sum \( w_N = \sum_{i=1}^{N} w_i e_i \) is an element of the Hilbert space with norm

\[ \| w_N \|_N^2 = \sum_{i=1}^{N} |w_i|^2 \]

by Bessel’s identity. Expanding out

\[ T(w_N) = T(\sum_{i=1}^{N} w_i e_i) = \sum_{i=1}^{N} w_i T(e_i) = \sum_{i=1}^{N} |w_i|^2 \]

and from the continuity of \( T \),

\[ |T(w_N)| \leq C \| w_N \|_H \Rightarrow \| w_N \|_H^2 \leq C \| w_N \|_H \Rightarrow \| w_N \|^2 \leq C^2 \]

which is the desired inequality.

(2) Letting \( N \to \infty \) it follows that the infinite sum converges and

\[ \sum_{i} |w_i|^2 \leq C^2 \Rightarrow w = \sum_{i} w_i e_i \in H \]

since \( \| w_N - w \| \leq \sum_{j > N} |w_j|^2 \) tends to zero with \( N \).

(3) For any \( u \in H \) \( u_N = \sum_{i=1}^{N} \langle u, e_i \rangle e_i \) by the completeness of the \( \{ e_i \} \) so from the continuity of \( T \)

\[ T(u) = \lim_{N \to \infty} T(u_N) = \lim_{N \to \infty} \sum_{i=1}^{N} \langle u, e_i \rangle T(e_i) \]

\[ = \lim_{N \to \infty} \sum_{i=1}^{N} \langle u, w_i e_i \rangle = \lim_{N \to \infty} \langle u, w_N \rangle = \langle u, w \rangle \]

where the continuity of the inner product has been used. From this and Cauchy’s inequality it follows that \( \| T \| = \sup_{\| u \|_H = 1} | T(u) | \leq \| w \| \). The converse follows from the fact that \( T(w) = \| w \|_H^2 \).