WITH SOLUTIONS

For full marks, complete and precise answers should be given to each question but you are not required to prove major results.

1. Problem 1

Let $H$ be a separable (partly because that is mostly what I have been talking about) Hilbert space with inner product $(\cdot, \cdot)$ and norm $\| \cdot \|$. Say that a sequence $u_n$ in $H$ converges weakly if $(u_n, v)$ is Cauchy in $C$ for each $v \in H$.

1. Explain why the sequence $\|u_n\|_H$ is bounded.
   Solution: Each $u_n$ defines a continuous linear functional on $H$ by
   \[
   T_n(v) = (v, u_n), \quad \|T_n\| = \|u_n\|, \quad T_n : H \rightarrow C.
   \]
   For fixed $v$ the sequence $T_n(v)$ is Cauchy, and hence bounded, in $C$ so by the ‘Uniform Boundedness Principle’ the $\|T_n\|$ are bounded, hence $\|u_n\|$ is bounded in $R$.

2. Show that there exists an element $u \in H$ such that $(u_n, v) \rightarrow (u, v)$ for each $v \in H$.
   Solution: Since $(v, u_n)$ is Cauchy in $C$ for each fixed $v \in H$ it is convergent. Set
   \[
   Tv = \lim_{n \rightarrow \infty} (v, u_n) \text{ in } C.
   \]
   This is a linear map, since
   \[
   T(c_1 v_1 + c_2 v_2) = \lim_{n \rightarrow \infty} c_1 (v_1, u_n) + c_2 (v_2, u_n) = c_1 Tv_1 + c_2 Tv_2
   \]
   and is bounded since $|Tv| \leq C \|v\|$, $C = \sup_n \|u_n\|$. Thus, by Riesz’ theorem there exists $u \in H$ such that $Tv = (v, u)$. Then, by definition of $T$,
   \[
   (u_n, v) \rightarrow (u, v) \quad \forall \ v \in H.
   \]

3. If $e_i, i \in N$, is an orthonormal sequence, give, with justification, an example of a sequence $u_n$ which is not weakly convergent in $H$ but is such that $(u_n, e_j)$ converges for each $j$.
   Solution: One such example is $u_n = n e_n$. Certainly $(u_n, e_i) = 0$ for all $i > n$, so converges to 0. However, $\|u_n\|$ is not bounded, so the sequence cannot be weakly convergent by the first part above.

4. Show that if the $e_i$ form an orthonormal basis, $\|u_n\|$ is bounded and $(u_n, e_j)$ converges for each $j$ then $u_n$ converges weakly.
   Solution: By the assumption that $(u_n, e_j)$ converges for all $j$ it follows that $(u_n, v)$ converges as $n \rightarrow \infty$ for all $v$ which is a finite linear combination of the $e_i$. For general $v \in H$ the convergence of the Fourier-Bessell series for $v$ with respect to the orthonormal basis $e_j$
   \[
   v = \sum_{k} (v, e_k) e_k
   \]
shows that there is a sequence $v_k \to v$ where each $v_k$ is in the finite span of the $e_j$. Now, by Cauchy’s inequality

\[(A.6) \quad |(u_n, v) - (u_m, v)| \leq |(u_n v_k) - (u_m v_k)| + |(u_n, v - v_k)| + |(u_m, v - v_k)|.\]

Given $\epsilon > 0$ the boundedness of $\|u_n\|$ means that the last two terms can be arranged to be each less than $\epsilon/4$ by choosing $k$ sufficiently large. Having chosen $k$ the first term is less than $\epsilon/4$ if $n, m > N$ by the fact that $(u_n, v_k)$ converges as $n \to \infty$. Thus the sequence $(u_n, v)$ is Cauchy in $\mathbb{C}$ and hence convergent.

2. Problem 2

Suppose that $f \in L^1(0, 2\pi)$ is such that the constants

$$c_k = \int_{(0,2\pi)} f(x) e^{-ikx}, \; k \in \mathbb{Z},$$

satisfy

$$\sum_{k \in \mathbb{Z}} |c_k|^2 < \infty.$$

Show that $f \in L^2(0, 2\pi)$.

Solution. So, this was a good bit harder than I meant it to be — but still in principle solvable (even though no one quite got to the end).

First, (for half marks in fact!) we know that the $c_k$ exists, since $f \in L^1(0, 2\pi)$ and $e^{-ikx}$ is continuous so $f e^{-ikx} \in L^1(0, 2\pi)$ and then the condition $\sum |c_k|^2 < \infty$ implies that the Fourier series does converge in $L^2(0, 2\pi)$ so there is a function

\[(A.1) \quad g = \frac{1}{2\pi} \sum_{k \in \mathbb{C}} c_k e^{ikx}.\]

Now, what we want to show is that $f = g$ a.e. since then $f \in L^2(0, 2\pi)$.

Set $h = f - g \in L^1(0, 2\pi)$ since $L^2(0, 2\pi) \subset L^1(0, 2\pi)$. It follows from (A.1) that $f$ and $g$ have the same Fourier coefficients, and hence that

\[(A.2) \quad \int_{(0,2\pi)} h(x) e^{ikx} = 0 \quad \forall \; k \in \mathbb{Z}.\]

So, we need to show that this implies that $h = 0$ a.e. Now, we can recall from class that we showed (in the proof of the completeness of the Fourier basis of $L^2$) that these exponentials are dense, in the supremum norm, in continuous functions which vanish near the ends of the interval. Thus, by continuity of the integral we know that

\[(A.3) \quad \int_{(0,2\pi)} h g = 0\]

for all such continuous functions $g$. We also showed at some point that we can find such a sequence of continuous functions $g_n$ to approximate the characteristic function of any interval $\chi_I$. It is not true that $g_n \to \chi_I$ uniformly, but for any integrable function $h$, $hg_n \to h\chi_I$ in $L^1$. So, the upshot of this is that we know a bit more than (A.3), namely we know that

\[(A.4) \quad \int_{(0,2\pi)} h g = 0 \quad \forall \text{ step functions } g.\]
So, now the trick is to show that (A.4) implies that \( h = 0 \) almost everywhere. Well, this would follow if we know that \( \int_{(0,2\pi)} \lvert h \rvert = 0 \), so let’s aim for that. Here is the trick. Since \( g \in L^1 \) we know that there is a sequence (the partial sums of an absolutely convergent series) of step functions \( h_n \) such that \( h_n \to g \) both in \( L^1(0,2\pi) \) and almost everywhere and also \( |h_n| \to |h| \) in both these senses. Now, consider the functions

\[
(A.5) \quad s_n(x) = \begin{cases} 
0 & \text{if } h_n(x) = 0 \\
\frac{h_n(x)}{|h_n(x)|} & \text{otherwise.}
\end{cases}
\]

Clearly \( s_n \) is a sequence of step functions, bounded (in absolute value by 1 in fact) and such that \( s_nh_n = |h_n| \). Now, write out the wonderful identity

\[
(A.6) \quad |h(x)| = |h(x)| - |h_n(x)| + s_n(x)(h_n(x) - h(x)) + s_n(x)h(x).
\]

Integrate this identity and then apply the triangle inequality to conclude that

\[
(A.7) \quad \int_{(0,2\pi)} |h| = \int_{(0,2\pi)} (|h(x)| - |h_n(x)|) + \int_{(0,2\pi)} s_n(x)(h_n - h) \\
\leq \int_{(0,2\pi)} (||h(x)| - |h_n(x)||) + \int_{(0,2\pi)} |h_n - h| \to 0 \text{ as } n \to \infty.
\]

Here on the first line we have used (A.4) to see that the third term on the right in (A.6) integrates to zero. Then the fact that \( |s_n| \leq 1 \) and the convergence properties.

Thus in fact \( h = 0 \) a.e. so indeed \( f = g \) and \( f \in L^2(0,2\pi) \). Piece of cake, right! Mia culpa.

3. Problem 3

Consider the two spaces of sequences

\[
h_{\pm 2} = \{ c : \mathbb{N} \to \mathbb{C}; \sum_{j=1}^{\infty} j^{\pm 2} |c_j|^2 < \infty \}.
\]

Show that both \( h_{\pm 2} \) are Hilbert spaces and that any linear functional satisfying

\[
T : h_2 \to \mathbb{C}, \quad |Tc| \leq C \|c\|_{h_2}
\]

for some constant \( C \) is of the form

\[
Tc = \sum_{j=1}^{\infty} c_j d_i
\]

where \( d : \mathbb{N} \to \mathbb{C} \) is an element of \( h_{-2} \).

Solution: Many of you hammered this out by parallel with \( l^2 \). This is fine, but to prove that \( h_{\pm 2} \) are Hilbert spaces we can actually use \( l^2 \) itself. Thus, consider the maps on complex sequences

\[
(A.1) \quad (T^\pm c)_j = c_j j^{\pm 2}.
\]

Without knowing anything about \( h_{\pm 2} \) this is a bijection between the sequences in \( h_{\pm 2} \) and those in \( l^2 \) which takes the norm

\[
(A.2) \quad \|c\|_{h_{\pm 2}} = \|Tc\|_{l^2}.
\]
It is also a linear map, so it follows that \( h_\pm \) are linear, and that they are indeed Hilbert spaces with \( T^\pm \) isometric isomorphisms onto \( l^2 \); The inner products on \( h_{\pm2} \) are then

\[
(c, d)_{h_{\pm2}} = \sum_{j=1}^{\infty} j^{\pm1} c_j \overline{d}_j.
\]

Don’t feel bad if you wrote it all out, it is good for you!

Now, once we know that \( h_2 \) is a Hilbert space we can apply Riesz’ theorem to see that any continuous linear functional \( T : h_2 \to \mathbb{C}, |Tc| \leq C \|c\|_{h_2} \) is of the form

\[
Tc = (c, d')_{h_2} = \sum_{j=1}^{\infty} j^4 c_j \overline{d}_j', \quad d' \in h_2.
\]

Now, if \( d' \in h_2 \) then \( d_j = j^4 d'_j \) defines a sequence in \( h_{-2} \). Namely,

\[
\sum_j j^{-4} |d_j|^2 = \sum_j j^4 |d'_j|^2 < \infty.
\]

Inserting this in (A.4) we find that

\[
Tc = \sum_{j=1}^{\infty} c_j d_j, \quad d \in h_{-2}.
\]