Lecture 13. Thursday, Mar 19: Baire’s theorem

Note from Lecture 9, modified and considerably extended.

**Theorem 8** (Baire). If \( M \) is a non-empty complete metric space and \( C_n \subset M, n \in \mathbb{N} \), are closed subsets such that
\[
(13.1) \quad M = \bigcup_n C_n
\]
then at least one of the \( C_n \)'s has an interior point.

**Proof.** So, choose \( p_1 \notin C_1 \), which must exist since otherwise \( C_1 \) contains an open ball. Since \( C_1 \) is closed there exists \( \epsilon_1 > 0 \) such that \( B(p_1, \epsilon_1) \cap C_1 = \emptyset \). Next choose \( p_2 \in B(p_1, \epsilon_1/3) \) which is not in \( C_2 \), which is possible since otherwise \( B(p_1, \epsilon_1/3) \subset C_2 \), and \( \epsilon_2 > 0, \epsilon_2 < \epsilon_1/3 \) such that \( B(p_2, \epsilon_2) \cap C_2 = \emptyset \). So we have used both the fact that \( C_2 \) has empty interior and the fact that it is closed. Now, proceed, inductively. Assume that there is a finite sequence \( p_i, i = 1, \ldots, k \) and positive numbers \( 0 < \epsilon_k < \epsilon_{k-1}/3 < \epsilon_{k-2}/3^2 < \cdots < \epsilon_1/3^{k-1} < 3^{-k} \) such that \( p_j \in B(p_{j-1}, \epsilon_{j-1}/3) \) and \( B(p_j, \epsilon_j) \cap C_j = \emptyset \). Then we can add another \( p_{k+1} \) by using the properties of \( C_k \) – it has non-empty interior so there is some point in \( B(p_k, \epsilon_k/3) \) which is not in \( C_{k+1} \) and then \( B(p_{k+1}, \epsilon_k/3) \cap C_{k+1} = \emptyset \) where \( \epsilon_k+1 > 0 \) but \( \epsilon_k+1 < \epsilon_k/3 \). Thus, we have construct and infinite sequence \( \{p_k\} \) in \( M \). Since \( d(p_{k+1}, p_k) < \epsilon_k/3 \) this is a Cauchy sequence. In fact
\[
(13.2) \quad d(p_k, p_{k+l}) < \epsilon_k/3 + \cdots + \epsilon_{k+l-1}/3 < 3^{-k} < 2\epsilon_k/3
\]
for all \( l > 0 \), and this tends to zero as \( k \to \infty \).

Since \( M \) is complete this sequence converges. From (13.2) the limit, \( q \in M \) must lie in the closure of \( B(p_k, 2\epsilon_k/3) \) for every \( k \). Hence \( q \notin C_k \) for any \( k \) which contradicts (13.1).

Thus, at least one of the \( C_n \) must have non-empty interior. \( \square \)

One application of this is often called the *uniform boundedness principle*, I will just call it:

**Theorem 9** (Uniform boundedness). Let \( B \) be a Banach space and suppose that \( T_n \) is a sequence of bounded (i.e. continuous) linear operators \( T_n : B \to V \) where \( V \) is a normed space. Suppose that for each \( b \in B \) the set \( \{T_n(b)\} \subset V \) is bounded (in norm of course) then \( \sup_n \|T_n\| < \infty \).

**Proof.** This follows from a pretty direct application of Baire’s theorem to \( B \). Consider the sets
\[
(13.3) \quad S_p = \{b \in B, \|b\| \leq 1, \|T_n b\|_V \leq p, \forall n\}, \quad p \in \mathbb{N}.
\]
Each \( S_p \) is closed because \( T_n \) is continuous, so if \( b_k \to b \) is a convergent sequence then \( \|b\| \leq 1 \) and \( \|T_n(p)\| \leq p \). The union if the \( S_p \) is the whole of the closed ball of radius one around the origin in \( B \):
\[
(13.4) \quad \{b \in B; d(b, 0) \leq 1\} = \bigcup_p S_p
\]
because of the assumption of ‘pointwise boundedness’ – each \( b \) with \( \|b\| \leq 1 \) must be in one of the \( S_p \)’s.
So, by Baire’s theorem one of the sets $S_p$ has non-empty interior. This means that for some $p$, some $v \in S_p$, and some $\delta > 0$,
\[
(13.5) \quad w \in B, \|w\|_B \leq \delta \implies \|T_n(v + w)\|_V \leq p \forall n.
\]

Moving $v$ to $(1 - \delta/2)v$ and halving $\delta$ as necessary it follows that this ball $B(v, \delta)$ is contained in the open ball around the origin of radius 1. Thus, using the triangle inequality, and the fact that $\|T_n(v)\|_V \leq p$ this implies
\[
(13.6) \quad w \in B, \|w\|_B \leq \delta \implies \|T_n(w)\|_V \leq 2p \implies \|T_n\| \leq 2p/\delta
\]
since the norm of the operator is $\sup \{\|T_n\|_V : \|w\|_B = 1\}$ it follows that the norms are uniformly bounded:
\[
(13.7) \quad \|T_n\| \leq 2p/\delta
\]
as claimed. \hfill \square

One immediate consequence of this is that, as I mentioned in last lecture, it is not necessary to assume that a weakly convergent sequence in a Hilbert space is norm bounded.

**Corollary 2.** If $u_n \in H$ is a sequence in a Hilbert space and for all $v \in H$
\[
(13.8) \quad (u_n, v) \rightarrow F(v) \text{ converges in } C
\]
then $\|u_n\|_H$ is bounded and there exists $w \in H$ such that $u_n \rightharpoonup w$ (converges weakly).

**Proof.** Well, a corollary really should not need a proof but still I will give one since maybe it is a bit more than a corollary.

Apply the Uniform Boundedness Theorem to the continuous functionals
\[
(13.9) \quad T_n(u) = (u, u_n), \quad T_n : H \rightarrow C
\]
where we reverse the order to make them linear rather than anti-linear. Thus, each set $\{T_n(u)\}$ is bounded in $C$ since it is convergent. It follows that there is a bound
\[
(13.10) \quad \|T_n\| \leq C.
\]
However, the norm is just $\|T_n\| = \|u_n\|_H$ so the sequence must be bounded in $H$. Define $T : H \rightarrow C$ as the limit for each $u$:
\[
(13.11) \quad T(u) = \lim_{n \rightarrow \infty} T_n(u) = \lim_{n \rightarrow \infty} (u, u_n).
\]
This exists for each $u$ by hypothesis. It is a linear map $a_n$ from $(13.10)$ it is bounded, $\|T\| \leq C$. Thus by the Riesz Representation theorem, there exists $w \in H$ such that
\[
(13.12) \quad T(u) = (u, w) \quad \forall u \in H.
\]
Thus $(u_n, u) \rightarrow (w, u)$ for all $u \in H$ so $u_n \rightharpoonup w$ as claimed. \hfill \square

The second major application of Baire’s theorem is to

**Theorem 10 (Open Mapping).** If $T : B_1 \rightarrow B_2$ is a bounded and surjective linear map between two Banach spaces then $T$ is open:
\[
(13.13) \quad T(O) \subset B_2 \text{ is open if } O \subset B_1 \text{ is open.}
\]

This is ‘wrong way continuity’ and as such can be used to prove the continuity of inverse maps as we shall see. The proof uses Baire’s theorem but then another similar sort of argument is needed. I did not finish the second argument in the lecture.
Proof. (1) The first part, of the proof, using Baire’s theorem shows that the closure of the image, so in $B_2$, of an open ball around the origin in $B_1$, has 0 as an interior point – i.e. it contains an open ball around the origin in $B_2$. To see this we apply Baire’s theorem to the sets

$$C_p = \text{cl}_{B_2} T(B(0, p))$$

the closure of the image of the ball in $B_1$ or radius $p$. We need to take the closure since the sets in Baire’s theorem are closed, but even before doing that we know that

$$B_2 = \sum_p T(B(0, p))$$

since that is what surjectivity means – every point is the image of something. Thus one of the closed sets $C_p$ has an interior point, $v$. Since $T$ is surjective, $v = Tu$ for some $u \in B_1$. The sets $C_p$ increase with $p$ so we can take a larger $p$ and $v$ is still an interior point, from which it follows that $0 = v - Tu$ is an interior point as well. Thus indeed

$$C_p \supset B(0, \delta)$$

for some $\delta > 0$.

(2) Having applied Baire’s theorem, consider now what (13.16) means. It follows that each $v \in B_2$ with $\|v\| < \delta$ is the limit of a sequence $Tu_n$ where $\|u_n\| \leq p$. What we want to arrange is that this sequence converges. Note that we can scale the norm of $v$ using the linearity of $T$. Thus, for a general $v \in B_2$ we can apply (13.16) to $v' = \delta v/2\|v\|$ to see that $Tu'_n \to v'$ where $\|u'_n\| \leq p$. Then $u_n = \|v\|u'_n/\delta$ satisfies $Tu_n \to v$, $\|u_n\| \leq 2p\|v\|/\delta$. 

To simplify the arithmetic, let me replace $T$ by $cT$ where $c = p/2\delta$. This means that for each $v \in B_2$ there is a sequence $u_n$ in $B_1$ with $\|u_n\| \leq \|v\|$ and $Tu_n \to v$.

Now, we can stop before we get to the limit of the sequence and get as close to $v$ as we want. This means that

$$\text{For each } v \in B_2, \exists u \in B_1, \|u\| < \|v\|, \|v - Tu\| \leq \frac{1}{2}\|v\|.$$
So finally we have shown that each \( w \in B(0, 1) \) in \( B_2 \) is in the image of \( B(0, 2) \) in \( B_1 \). Going back to the unscaled \( T \) it follows that for some \( \delta > 0 \),

\[
\text{(13.20)} \quad B(0, \delta) \subset T(B(0, 1)).
\]

(3) It follows of course that the image \( T(O) \) of any open set is open, since if \( w \in T(O) \) then \( w = Tu \) for some \( u \in O \) and hence \( B(w, \epsilon \delta) \) is contained in the image of \( u + B(0, \epsilon) \subset O \) for \( \epsilon > 0 \) sufficiently small.

\[
\text{□}
\]

So, as I did not quite finish the proof in lecture. However at the very end I mention the two most important applications of of this ‘Open Mapping Theorem’. Namely:

**Corollary 3.** If \( T : B_1 \rightarrow B_2 \) is a bounded linear map between Banach spaces which is 1-1 and onto, i.e. is a bijection, then it is a homeomorphism – meaning its inverse, which is necessarily linear, is also bounded.

**Proof.** The only confusing thing is the notation. Note that \( T^{-1} \) is used to denote the inverse maps on sets. So, the inverse of \( T \), let’s call it \( S : B_2 \rightarrow B_1 \) is certainly linear. If \( O \subset B_1 \) is open then \( S^{-1}(O) = T(O) \) is open by the Open Mapping theorem, so \( S \) is continuous.

The second application is

**Theorem 11** (Closed Graph). If \( T : B_1 \rightarrow B_2 \) is a linear map between Banach spaces then it is bounded if and only if its graph

\[
\text{(13.21)} \quad \text{Gr}(T) = \{ (u, v) \in B_1 \times B_2; u_2 = Tu_1 \}
\]

is a closed subset of the Banach space \( B_1 \times B_2 \).

Have we actually covered the product of Banach spaces explicitly? If not, think about it for a minute or two!

**Proof.** Suppose first that \( T \) is bounded, i.e. continuous. A sequence \( (u_n, v_n) \in B_1 \times B_2 \) is in \( \text{Gr}(T) \) if and only if \( v_n = Tu_n \). So, if it converges, then \( u_n \rightarrow u \) and \( v_n = Tu_n \rightarrow Tu \) by the continuity of \( T \), so the limit is in \( \text{Gr}(T) \) which is therefore closed.

Conversely, suppose the graph is closed. Given the graph we can reconstruct the map it comes from (whether linear or not) from a little diagram. Form \( B_1 \times B_2 \) consider the two projections, \( \pi_1(u, v) = u \) and \( \pi_2(u, v) = v \). Both of them are continuous by inspection and we can restrict them to \( \text{Gr}(T) \subset B_1 \times B_2 \) to get

\[
\text{(13.22)} \quad \begin{array}{c}
\text{Gr}(T) \\
B_1 \\
\downarrow \pi_1 \\
\downarrow T \\
B_2
\end{array}
\]

This little diagram commutes. Indeed there are two ways to map a point \( (u, v) \in \text{Gr}(T) \) to \( B_2 \), either directly, sending it to \( v \) or first sending it \( u \in B_1 \) and then to \( Tu \). Since \( v = Tu \) these are the same.

Now, \( \text{Gr}(T) \subset B_1 \times B_2 \) is a closed subspace, so it too is a Banach space and \( \pi_1 \) and \( \pi_2 \) remain continuous when restricted to it. The map \( \pi_1 \) is 1-1 and onto, because each \( u \) occurs as the first element of precisely one pair, namely \( (u, Tu) \in \text{Gr}(T) \).
Thus the Corollary above applies to $\pi_1$ to show that its inverse, $S$ is continuous. But then $T = \pi_2 \circ S$, from the commutativity, is also continuous proving the theorem. $\square$