Problem Set 8

1. Let $f \in L^1(\mathbb{R}/2\pi\mathbb{Z})$, and let $\sigma_N$ denote the Cesaro mean of its Fourier series. Prove that if $f$ has a left and right limit at $x$, then

$$\sigma_N(x) \to (f(x^+) + f(x^-))/2 \quad \text{as} \quad N \to \infty$$

(You may use the formula from lecture for $F_N$ such that $\sigma_N(x) = f * F_N(x)$.)

Hint: Formulate and prove a variant of the “approximate identity” lemma, with stronger hypotheses on $K_N$ in exchange for weaker properties of $f$, and confirm the stronger properties of $F_N$ that you need.

2. Consider the Fourier series for $f$ from 2a PS7 at $x = 0$ and $x = \pi$; $g$ from 2b at $x = 0$; $h$ from 2c at $x = \pi/2$. What are the consequences of the theorems in problems 3 PS7 and problem 1 above at these points?

3. Let $R_N$ denote the $2^N$ dyadic intervals of $[0, 1)$ of length $2^{-N}$, that is,

$$R_N = \{I = [(k - 1)/2^N, k/2^N) : k = 1, 2, \ldots, 2^N\}$$

Consider

$$V_N = \text{span} \{1_I : I \in R_N\}$$

Let $P_N : L^2([0, 1]) \to V_N$ be the orthogonal projection onto $V_N$, that is, the mapping such that $P_N f = f$ for all $f \in V_N$ and $P_N f \perp (f - P_N f)$ for all $f \in L^2([0, 1])$.

a) Find the formula for $a_I$ (in terms of $I$ and $f$) such that

$$P_N f = \sum_{I \in R_N} a_I 1_I$$

and show that $P_N f$ tends uniformly (on $[0, 1]$) to $f$ for all $f \in C([0, 1])$.

b) Let $1 \leq p < \infty$. Show that $P_N f$ tends to $f$ in $L^p([0, 1])$ for every $f \in L^p([0, 1])$.

c) For $f \in L^1([0, 1])$, find the formula for $P_0 f$ and $P_{N+1} f - P_N f$ in terms of $\langle f, H_{n,k} \rangle$ and $H_{n,k}$, the Haar functions defined in AG §3.3/11, pp. 136–137. Warning: identify the misprint in part (a) p. 137. Deduce that the Haar functions form a complete orthonormal system of $L^2([0, 1])$.

4. a) Do AG §3.3/9, p. 136 (Gram-Schmidt process).

b) Use power series to show that every function $e^{inx}$ can be uniformly approximated on $[-\pi, \pi]$ by polynomials (ordinary polynomials in $x$).
c) Deduce from (b) that polynomials are dense in $L^2([-\pi, \pi])$.

d) Denote by $\psi_0, \psi_1, \ldots$, the functions obtained from the Gram-Schmidt process applied to the polynomials $f_0(x) = 1, f_1(x) = x, f_2(x) = x^2, \ldots$. Show that these form an orthonormal basis of $L^2([-\pi, \pi])$ and compute the first three. (The answers on $[-1, 1]$ are listed in AG §3.3/10 p. 136.)

Show further that the degree of $\psi_n$ is $n$ and that $\psi_n$ is even if $n$ is even and odd if $n$ is odd.

e) Show by integration by parts that

$$R_n(x) = \frac{d^n}{dx^n}(x^2 - 1)^n$$

is orthogonal to $1, x, \ldots, x^{n-1}$ in $L^2([-1,1])$ and $R_n(1) = 2^n n!$. (Hint: $x^2 - 1 = (x-1)(x+1)$.)

f) The Legendre polynomials are defined as the polynomials

$$P_n(x) = \frac{R_n(x)}{2^n n!}.$$ 

In other words, they are normalized so that $P_n(1) = 1$. Show how your formulas for $\psi_n$, $n = 0, 1, 2$ in (c) match this formula for $P_n$.

5. Define the Laplace operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ on the $(x,y)$-plane.

a) Show that in polar coordinates $(x = r \cos \theta, y = r \sin \theta),

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

b) Let $f \in C(\mathbb{R}/2\pi \mathbb{Z}).$ Define $u$ in polar coordinates by

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}(n) e^{i n \theta}, \quad 0 \leq r < 1$$

Express $u$ as a series in $z = x + iy$ and $\bar{z} = x - iy$. Confirm that $u$ is infinitely differentiable in $x^2 + y^2 < 1$ and that $\Delta u = 0$ for $0 \leq r < 1$. Solutions to $\Delta u = 0$ are known as harmonic functions.

\footnote{The functions $\varphi_n$ of AG §3.3/10 p. 136 indexed starting from $n = 1$ and with the normalization that the $L^2$ norm on $[-1, 1]$ is 1 differ from the customary notation for Legendre polynomials $P_n$. Further properties (not assigned) are as follows.

$$\sum_{n=0}^{\infty} P_n(x) z^n = \frac{1}{\sqrt{1 - 2xz + z^2}} \quad \text{(generating function)}$$

Recurrence formula and $L^2$ norm:

$$(n - 1)P_n(x) = (2n - 1) x P_{n-1}(x) - n P_{n-2}; \quad \int_{-1}^{1} P_n(x)^2 dx = 2/(2n + 1).$$}
Remark. One should think of $f(\theta)$ as a function on the unit circle $\{e^{i\theta} : \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$ in the complex plane and $u$ is a function of $z = re^{i\theta}$ in the unit disk. Then $u$ is the harmonic function with boundary values $f$, as we now prove.

c) Compute the Poisson kernel $P_r$ satisfying

$$u(r, \theta) = f * P_r(\theta)$$

Prove that if $f \in C(\mathbb{R}/2\pi\mathbb{Z})$, then

$$\max_{\theta} |u(r, \theta) - f(\theta)| \to 0 \quad \text{as } r \to 1^-$$

If $f \in L^1(\mathbb{R}/2\pi\mathbb{Z})$, then

$$\lim_{r \to 1^-} \int_{[-\pi,\pi]} |f * P_r(\theta) - f(\theta)| \, d\theta = 0$$

d) (Extra credit) Prove that if $f$ is continuous, then $u$ extends to a continuous function on the closed unit disk. In other words,

$$u(r_j, \theta_j) \to f(\theta)$$

whenever $r_j \to 1^-$ and $\theta_j \to \theta$.

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2Given that $u$ is continuous in the closed disk, one can prove that $u$ is unique using what is known as the maximum principle. The maximum principle (for the disk) says that if $v(z)$ is real-valued and continuous in $|z| \leq 1$ and harmonic in $|z| < 1$, then

$$\max_{|z| \leq 1} v(z) \leq \max_{|z| = 1} v(z)$$

Let $v$ be $\pm$ the difference of any two real-valued harmonic functions with the same boundary values, then by the maximum principle, $v = 0$ and the two functions are the same. Using uniqueness for continuous boundary values, one can deduce uniqueness of $u$ with boundary values in the $L^1$ sense stated above.