The Fourier transform on \( \mathbb{R} \) is defined for all \( f \in L^1(\mathbb{R}) \) by \( \hat{f}(t) = \int f(x)e^{-itx}dx \). Denoting \( G(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \), we have
\[
\hat{G}(t) = e^{-t^2/2}
\]

**Main Approximate Identity Lemma.** Let \( K \in L^1(\mathbb{R}) \) satisfy
\[
\int \mathbb{R} K(x) \, dx = 1
\]
and denote \( K_a(x) = \frac{1}{a}K(x/a), a > 0 \). Then for every \( x \in \mathbb{R} \) and \( f \in C_0(\mathbb{R}) \),
\[
\lim_{a \to 0^+} f * K_a(x) = f(x)
\]

**Theorem 1 (Fourier inversion on \( \mathcal{S} \)).** The mappings \( T_1 \) and \( T_2 \) defined for \( f \) and \( g \) in \( \mathcal{S}(\mathbb{R}) \) by the Riemann integrals
\[
(T_1f)(t) = \int \mathbb{R} f(x)e^{-itx}dx; \quad (T_2g)(x) = \frac{1}{2\pi} \int \mathbb{R} g(t)e^{itx}dt
\]
send the Schwartz class \( \mathcal{S} \) to itself. Moreover, the compositions \( T_2T_1 \) and \( T_1T_2 \) are both the identity mapping on \( \mathcal{S} \).

**Theorem 2 (Plancherel).**
\[a) \quad \|T_1f\|^2 = 2\pi\|f\|^2; \quad 2\pi\|T_2g\|^2 = \|g\|^2\]
where
\[\|f\|^2 = \int \mathbb{R} |f(x)|^2 \, dx\]
\[b) \quad T_1 \text{ and } T_2 \text{ have unique extensions from } \mathcal{S} \text{ to continuous mappings from } L^2(\mathbb{R}) \text{ to itself, } T_1T_2 \text{ and } T_2T_1 \text{ are the identity mapping on } L^2(\mathbb{R}) \text{ and the properties of part (a) are valid for all } f \text{ and } g \text{ in } L^2(\mathbb{R}).\]

**Theorem 3 (Fourier inversion with truncation).** Let \( f \in L^2(\mathbb{R}) \), and denote
\[
s_N(x) = \frac{1}{2\pi} \int_{-N}^{N} \hat{f}(\xi)e^{ixt} \, d\xi
\]
Then
\[
\lim_{N \to \infty} \int \mathbb{R} |s_N(x) - f(x)|^2 \, dx = 0
\]

**Proposition.** If \( f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \), then
\[
T_1f(t) = \int \mathbb{R} f(x)e^{-itx}dx \quad T_2g(x) = \frac{1}{2\pi} \int \mathbb{R} g(t)e^{itx}dt
\]

**Theorem 4.** Let
\[
f * g(x) = \int \mathbb{R} f(x-y)g(y)dy
\]
If \( f \in L^1(\mathbb{R}) \) and \( g \in L^1(\mathbb{R}) \) or (and this requires more work) if \( f \in L^2(\mathbb{R}) \) and \( g \in L^2(\mathbb{R}) \), then
\[
(\hat{f * g})(t) = \hat{f}(t)\hat{g}(t)
\]
Review Problems

1.

a) Find the Fourier series of the function

\[ f(x) = \begin{cases} 
1, & 0 < x < \pi; \\
0, & -\pi < x < 0.
\end{cases} \]

extended periodically with period \(2\pi\). Pay attention to three cases \(n = 0\) and \(n \neq 0\) odd and even, separately.

b) Express your series with real numbers, sines and cosines.

c) At which points \(x\) does the series converge and to what value? Explain with statements of theorems.

2. Suppose that \(f \in L^2(\mathbb{R}/2\pi\mathbb{Z})\) takes the form

\[ f(\theta) = \sum_{n=1}^{\infty} a_n e^{in\theta} \]

Recall that if \(z = re^{i\theta} = x + iy\),

\[ F(z) = \sum_{n=1}^{\infty} r^n a_n e^{in\theta} \]

is a harmonic (and even analytic) function in \(|z| < 1\).

a) Why does the series for \(F(z)\) converge for \(|z| < 1\)?

b) Let \(f_r(\theta) = F(re^{i\theta})\), the values of \(F\) on the circle of radius \(r\). Calculate \(\|f_r - f\|_2\) in terms \(r\) and \(a_n\), and show that \(F\) takes on the boundary values in the sense that

\[ \lim_{r \to 1^-} \|f_r - f\|_2 = 0 \]

c) Evaluate the integral

\[ \int_{|z| < 1} \left| (\partial/\partial r)F(z) \right|^2 (1 - |z|) \, dx \, dy \]

in terms of the coefficients \(a_n\). Explain at an appropriate point before, during or after the computation, why the integral is finite.

3. Fourier inversion on the Schwartz class \(S(\mathbb{R})\). (Approximate identity Lemma and Theorem 1 above.)

a) Recall that \(C_0(\mathbb{R})\) is defined as the class of continuous functions on \(\mathbb{R}\) that tend to zero at \(\pm\infty\). Show that if \(K \in L^1(\mathbb{R})\) and

\[ \int_{\mathbb{R}} K(x) \, dx = 1; \quad K_a(x) = \frac{1}{a} K(x/a), \quad a > 0 \]

then

\[ \lim_{a \to 0} f * K_a(x) = f(x) \]
for every $x \in \mathbb{R}$ and every $f \in C_0(\mathbb{R})$. Make use in your proof of the quantities

$$Q = \int_{\mathbb{R}} |K(x)| \, dx; \quad M = \max_{x \in \mathbb{R}} |f(x)|$$

and the modulus of continuity of $f$,

$$\omega(r) = \max_{x \in \mathbb{R}; \, |y| \leq r} |f(x + y) - f(x)|$$

b) Show that for every $f \in \mathcal{S}$, $f(0) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t) \, dt$. You may assume without proof that for every $f$ and $g$ in $\mathcal{S}$, $\hat{f}$ and $\hat{g}$ belong to $\mathcal{S}$ and

$$\int_{\mathbb{R}} f(y) \hat{g}(y) \, dy = \int_{\mathbb{R}} \hat{f}(t) g(t) \, dt$$

c) Deduce the Fourier inversion formula (formula for $f(x)$ in terms of $\hat{f}$) for $f \in \mathcal{S}$.

4. Fourier inversion formula on $L^2(\mathbb{R})$ (Proof of Theorem 3 and the proposition above.)

a) For $f \in L^2(\mathbb{R})$ and denote

$$s_N(x) = \frac{1}{2\pi} \int_{-N}^{N} \hat{f}(t) e^{ixt} \, dt$$

Explain why the integral defining $s_N(x)$ converges and why $s_N$ is continuous.

b) Prove that if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then

$$(T_1 f)(t) = \int_{\mathbb{R}} f(x) e^{-itx} \, dx$$

following the three steps with *'s below.

You may assume that for any $f \in L^1 \cap L^2$, there is a sequence of functions $f_k \in \mathcal{S}$ such that $\|f - f_k\|_{L^1} + \|f - f_k\|_{L^2} \to 0$ as $k \to \infty$. Define

$$\varphi_k(t) = \int_{\mathbb{R}} f_k(x) e^{-itx} \, dx; \quad \varphi(t) = \int_{\mathbb{R}} f(x) e^{-itx} \, dx$$

* Show that $\varphi_k(t)$ tends to $\varphi(t)$ for each $t$ as $k \to \infty$.

* Show that $\|\varphi_k - T_1 f\|_{L^2}$ tends to 0 as $k \to \infty$.

* Deduce that $\varphi(t) = (T_1 f)(t)$ (This equality holds in what sense?) Hint: Fatou’s lemma leads to the fastest proof, but you may use other methods.

c) Deduce that

$$\lim_{N \to \infty} \int_{\mathbb{R}} |f(x) - s_N(x)|^2 \, dx = 0$$

using the statement analogous to part (b) for $T_2$ and the other theorems on the page of theorems as necessary.

5. Poisson summation formula. Let $\varphi \in \mathcal{S}(\mathbb{R})$. Show that

$$\sum_{n \in \mathbb{Z}} \varphi(2\pi n) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \hat{\varphi}(k)$$
by calculating the Fourier series of 

\[ F(x) = \sum_{n \in \mathbb{Z}} \varphi(x - 2\pi n) \]

in two ways.

6. Recall that

\[ P_y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-y|\xi|} e^{ix\xi} \, d\xi = \frac{1}{\pi} \frac{y}{x^2 + y^2} \]

satisfies for all \( x \in \mathbb{R} \) and all \( y > 0 \),

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) P_y(x) = 0, \]

In other words, \( P_y(x) \) is harmonic in the upper half-plane \( \{ (x, y) \in \mathbb{R}^2 : y > 0 \} \) and for \( f \in L^1(\mathbb{R}) \),

\[ u(x, y) = P_y * f(x) \]

is harmonic in the upper half plane \( y > 0 \).

If \( f \in S(\mathbb{R}) \), use the Fourier transform to calculate

\[ \int_0^{\infty} \int_{-\infty}^{\infty} |\nabla u(x, y)|^2 y \, dx \, dy \]

in terms of \( f \). (Either before during or after the calculation, justify all the exchanges of integrals/differentiation/limits.)