4. a) (T/F) If \( A_k \) are measurable subsets of \( \mathbb{R} \), then \( \lim_{N \to \infty} \mu \left( \bigcap_{k=1}^{N} A_k \right) = \mu \left( \bigcap_{k=1}^{\infty} A_k \right) \)

False. (Only works when one of the measures is finite.) Let \( A_k = [k, \infty) \), then the limit is infinity, whereas
\[
\emptyset = \bigcap_{k=1}^{\infty} A_k,
\]
so that the right side is zero.

b) (T/F) If \( f(x, y) \geq 0 \) is measurable on \( \mathbb{R} \times \mathbb{R} \), and \( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(x,y) d\mu(x) \right) d\mu(y) < \infty \), then \( \frac{xyf(x,y)}{x^2 + y^2} \) is integrable on \( \mathbb{R} \times \mathbb{R} \).

True. Note that \( \frac{xyf(x,y)}{x^2 + y^2} \) is measurable. By the version of Fubini’s theorem on a problem set, \( f \) is integrable on \( \mathbb{R}^2 \) with respect to \( \mu \times \mu \). Finally, because \( x^2 - 2xy + y^2 = (x-y)^2 \geq 0 \),
\[
\left| \frac{xy}{x^2 + y^2} \right| \leq \frac{1}{2}
\]
Therefore,
\[
\int_{\mathbb{R} \times \mathbb{R}} \frac{xyf(x,y)}{x^2 + y^2} d(\mu \times \mu) \leq \frac{1}{2} \int_{\mathbb{R} \times \mathbb{R}} f(x,y) d(\mu \times \mu) < \infty
\]
Thus the function is integrable.

5. If \( f_n \) is a sequence of measurable functions on \([0, 1]\) such that \( 0 \leq f_n(x) \leq 1 \). Then
\[
\limsup_{n \to \infty} \int_0^1 f_n(x) \, d\mu(x) \leq \int_0^1 \limsup_{n \to \infty} f_n(x) \, d\mu(x),
\]
This is proved by applying Fatou’s lemma to the functions \( g_n(x) = 1 - f_n(x) \). The inequality may be strict as in this example with LHS = 1/2; RHS = 1.

\[
f_{2n}(x) = \begin{cases} 0 & 0 \leq x \leq 1/2 ; \\ 1 & 1/2 < x \leq 1 \end{cases}; \quad f_{2n+1}(x) = \begin{cases} 1 & 0 \leq x \leq 1/2 \\ 0 & 1/2 < x \leq 1 \end{cases}
\]