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Introduction

Useful Formulas:

\[ \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \]

\[ \Gamma(x + 1) = x\Gamma(x) \]

\[ \Gamma(n) = (n - 1)! \]
More Useful Formulas:

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots
\]
Weierstrass’ Product Formula

Theorem (Gauss):

\[ \Gamma(x) = \lim_{{n \to \infty}} \frac{n^x n!}{x(x + 1) \cdots (x + n)} \]
Weierstrass’ Product Formula

\[ \Gamma(x) = e^{-Cx} \frac{1}{x} \prod_{i=1}^{\infty} \frac{e^{x/i}}{1 + x/i}, \]

where \( C = \lim_{n \to \infty} \left( \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \right) \)
Multiplication Formula

Gauss’ Multiplication Formula

\[
\frac{(2\pi)^{(p-1)/2}}{p^{x-1/2}} \Gamma(x) = \Gamma\left(\frac{x}{p}\right)\Gamma\left(\frac{x+1}{p}\right) \cdots \Gamma\left(\frac{x + p - 1}{p}\right)
\]

Legendre’s Relation, where \( p = 2 \)

\[
\frac{\sqrt{\pi}}{2^{x-1}} \Gamma(x) = \Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x + 1}{2}\right)
\]
Define:

\[ \phi(x) = \Gamma(x)\Gamma(1 - x) \sin \pi x, \]

then \( \phi(x + 1) = \phi(x) \).
Sine and Gamma Functions

Proof:
\[ \Gamma(-x + 1) = -x \Gamma(-x) \]
\[ \Gamma(-x) = \frac{\Gamma(-x + 1)}{-x} \]
\[ \phi(x + 1) = \Gamma(x + 1) \Gamma(-x) \sin(\pi(x + 1)) \]
\[ \phi(x + 1) = x \Gamma(x) \frac{\Gamma(-x + 1)}{-x} (-\sin \pi x) \]
\[ \phi(x + 1) = \Gamma(x) \Gamma(-x + 1) \sin \pi x = \phi(x) \]
\[ b2^{-x} \Gamma(x) = \Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x + 1}{2}\right) \]

\[ b2^{x-1} \Gamma(1 - x) = \Gamma\left(\frac{1-x}{2}\right)\Gamma\left(1 - \frac{x}{2}\right) \]
\[ \phi\left(\frac{x}{2}\right) \phi\left(\frac{x+1}{2}\right) = \Gamma\left(\frac{x}{2}\right) \Gamma\left(1 - \frac{x}{2}\right) \sin \frac{\pi x}{2} \Gamma\left(\frac{x+1}{2}\right) \Gamma\left(\frac{1-x}{2}\right) \cos \frac{\pi x}{2} \]

and

\[ b^2 = \frac{1}{4} \Gamma(x) \Gamma(1-x) \sin \pi x \]

and

\[ b^2 = \frac{1}{4} \phi(x) \]
$\phi(x) = \Gamma(x) \Gamma(1-x) \sin \pi x$

$$= \frac{\Gamma(1+x)}{x} \Gamma(1-x)(\pi x - \frac{\pi x^3}{3!} + \frac{\pi x^5}{5!} - \frac{\pi x^7}{7!} + \cdots)$$

$$= \Gamma(1+x) \Gamma(1-x)(\pi - \frac{\pi^3 x^2}{3!} + \frac{\pi^5 x^4}{5!} - \frac{\pi^7 x^6}{7!} + \cdots)$$

$\phi(0) = \pi$
Define $g(x)$ to be a periodic function, which is the second derivative of $\log(\phi(x))$.

It is bounded and the bound of $g(x)$ goes to 0, so $g(x) = 0$ and $\log(\phi(x))$ is linear.

Since $\log(\phi(x))$ is periodic, it must be constant.

Therefore $\phi(x)$ is constant and equals $\pi$ for all $x$. 
Since $\phi(x) = \pi$,

$$
\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}
$$

$$
\sin \pi x = \frac{\pi}{-x\Gamma(x)\Gamma(-x)}
$$
Sine and Gamma Functions

Sine Product Formula

\[
\sin \pi x = \pi x \prod_{i=1}^{\infty} \left(1 - \frac{x^2}{i^2}\right)
\]
Applications

Riemann Zeta Function:

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots \]

\[ \zeta(2) = \frac{\pi^2}{6} \]

\[ \zeta(4) = \frac{\pi^4}{90} \]
From zeta(2):

$$\sum_{i=1}^{\infty} \frac{1}{i^2 \pi^2} = \frac{1}{6}$$

$$\left(\sum_{i=1}^{\infty} \frac{1}{i^2 \pi^2}\right)^2 = \sum_{i=1}^{\infty} \frac{1}{i^4 \pi^4} + 2 \sum_{i \neq j} \frac{1}{i^2 j^2 \pi^4} = \frac{1}{36}$$
Applications

Using the Sine Product Formula:

\[
\frac{\sin x}{x} = (1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{4\pi^2})(1 - \frac{x^2}{9\pi^2}) \ldots
\]

The coefficient of \(x^4\):

\[
\sum_{i \neq j} \frac{1}{i^2 j^2 \pi^4} = \frac{1}{5!} = \frac{1}{120}
\]
Applications

\[
\left( \sum_{i=1}^{\infty} \frac{1}{i^2 \pi^2} \right)^2 = \sum_{i=1}^{\infty} \frac{1}{i^4 \pi^4} + 2 \sum_{i \neq j} \frac{1}{i^2 j^2 \pi^4}
\]

\[
\frac{1}{36} = \sum_{i=1}^{\infty} \frac{1}{i^4 \pi^4} + 2 \left( \frac{1}{120} \right)
\]

\[
\zeta(4) = \sum_{i=1}^{\infty} \frac{1}{i^4} = \frac{\pi^4}{90}
\]