The Golden Ratio

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Outline

- Geometric Definition
- Relation with Fibonacci Numbers
- Euclidean Geometric Construction
- Continuous Fraction Representation
Geometric Definition
(Mean and Extreme Ratio)

- $x$ satisfies
  \[ x^2 - x - 1 = 0 \]
- Golden ratio = positive root = $\tau = \frac{1 + \sqrt{5}}{2}$
- Negative root = $1 - \tau = \mu = \frac{1 - \sqrt{5}}{2}$
Relation with Fibonacci Numbers

- Binet’s Formula

\[ f_n = \frac{(\tau^n - \mu^n)}{\sqrt{5}} \]

- \[ \frac{f_{n+1}}{f_n} = \frac{\tau^{n+1} - \mu^{n+1}}{\tau^n - \mu^n} \]

- \[ \tau = \lim_{n \to \infty} \frac{f_{n+1}}{f_n} \text{ since } |\tau/\mu| > 1 \]
Geometric Construction

- Construct a right triangle with sides $\frac{1}{2}$ and 1
- Add the hypotenuse and shortest side

\[ \frac{1}{2} \quad \frac{1}{2} \sqrt{5} \]

\[ \frac{1}{2} \quad 1 \quad \frac{1}{2} \sqrt{5} \]
Continuous Fraction Representation

\[ \tau = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}} \]

\[ \tau = \lim_{n \to \infty} u_n, \text{ where } u_{n+1} = 1 + \frac{1}{u_n} \]

and \( u_1 = 1 \)

\[ \text{Let } u_n = \frac{a_{n+1}}{a_n} \text{ with } a_1 = a_2 = 1 \]

\[ \text{Recursion of } a_n \text{ is } a_{n+2} = a_{n+1} + a_n \]

\[ \{a_n\} = \text{Fibonacci numbers} \]
Infinite Resistor Network

- Each resistor has resistance $1\Omega$
- Total resistance = $r = ?$
- Recall
  - Total = $a + b$
  - Total = $\frac{ab}{a+b}$
Infinite Resistor Network (continued)

\[ r_{n+1} = 1 + \frac{1}{1 + \frac{1}{r_n}} \]

\[ r = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}} = \tau \]
Exercise on Continued Fractions

*(Young, Problem 9, page 156)*

- Find \( p = 2a + \frac{b}{2a + \frac{b}{2a + \cdots}} \)

  with \( a, b \) positive integers

- \( p = \lim_{n \to \infty} p_n \) with \( p_1 = 2a \) and

  \[
p_{n+1} = 2a + \frac{b}{p_n}
  \]

- Define

  \[
p_n = \frac{u_{n+1}}{u_n}, \quad u_1 = 1, \quad u_2 = 2a
  \]
Exercise on Continued Fractions (continued)

\[ u_{n+2} - 2au_{n+1} - bu_n = 0 \]

- Basis of solutions: \( u_n = \lambda^n \)
  \[ \lambda^2 - 2a\lambda - b = 0 \]

- \( \alpha = a + \sqrt{a^2 + b}, \beta = a - \sqrt{a^2 + b} \)

- Note \( |\beta| < a + \sqrt{a^2 + b} = \alpha \)

- General solution \( u_n = c\alpha^n + d\beta^n \)

\( u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \) (matching \( u_1 \) and \( u_2 \))
For any four consecutive Fibonacci numbers \( f_{n-1}, f_n, f_{n+1}, f_{n+2} \)
show that \( f_{n-1} f_{n+2} \) and \( 2 f_n f_{n+1} \)
form two shortest sides of a Pythagorean triangle.

Write \( f_n = b \) and \( f_{n+1} = a, \) \( a > b \)

\[
a - b, \ b, \ a, \ a + b
\]

\[
x = a^2 - b^2, \ y = 2ab
\]

\[
x^2 + y^2 = z^2, \ z = a^2 + b^2
\]
Hypotenuse \[ z = f_n^2 + f_{n+1}^2 \]

From previous class,
\[ f_n^2 + f_{n+1}^2 = f_{2n+1} \]

How is the area related to the original four numbers?
\[ A = \frac{xy}{2} = f_{n-1}f_n f_{n+1}f_{n+2} \]

Product of four consecutive Fibonacci numbers is the area of a Pythagorean triangle