Problems for 18.112 Mid 2 (Open Book)

Nov. 22, 2006

1. (15’) Evaluate
\[ \int_{\gamma} \frac{dz}{e^z - 1} \]
where \( \gamma \) is the circle \(|z| = 9\).

2. (30’) Let \( f(z) \) be analytic in the whole plane and assume that \( \frac{\text{Re}(f(z))}{z} \to 0 \) as \( z \to \infty \). Prove: \( f \) is a constant.
(Hint: Use formula (66) valid for \(|z| < R\):
\[ f(z) = \frac{1}{2\pi i} \int_{|\zeta| = R} \frac{\zeta + z}{\zeta - z} u(\zeta) \frac{d\zeta}{\zeta} + iC, \quad u = \text{Re}(x) \]
and estimate \( f'(z) \) carefully for \( z < \frac{R}{2} \) (Liouville). )

3. (25’) If \( f(z) \) is analytic for \(|z| < 1\) and
\[ |f(z)| \leq \frac{1}{1 - |z|^n}, \]
find the best estimate of \( f^{(n)}(0) \) that Cauchy’s formula will yield.
(Hint: Use Cauchy’s formula in each \(|z| \leq r, (r < 1)\).)

4. (30’) How many roots does the equation
\[ z^7 - 2z^5 + 6z^3 - z + 1 = 0 \]
have in the disk \(|z| < 1\)? How many roots are inside \(|z| = 2\)?
(Hint: Look for the biggest term when \(|z| = 1\), use Rouche.)