Lecture 13: The General Cauchy Theorem

(Replacing Text 137-148)

Here we shall give a brief proof of the general form of Cauchy’s Theorem.

**Definition 1** A closed curve $\gamma$ in an open set $\Omega$ is **homologous to 0** (written $\gamma \sim 0$) with respect to $\Omega$ if

$$n(\gamma, a) = 0 \quad \text{for all } a \notin \Omega.$$ 

**Definition 2** A region is **simply connected** if its complement with respect to the extended plane is connected.

**Remark:** If $\Omega$ is simply connected and $\gamma \subset \Omega$ a closed curve, then $\gamma \sim 0$ with respect to $\Omega$. In fact, $n(\gamma, z)$ is constant in each component of $\mathbb{C} - \gamma$, hence constant in $\mathbb{C} - \Omega$ and is 0 for $z$ sufficiently large.

**Theorem 1 (Cauchy’s Theorem)** If $f$ is analytic in an open set $\Omega$, then

$$\int_{\gamma} f(z) \, dz = 0$$

for every closed curve $\gamma \subset \Omega$ such that $\gamma \sim 0$.

In particular, if $\Omega$ is simply connected then $\int_{\gamma} f(z) \, dz = 0$ for every closed $\gamma \subset \Omega$.

We shall first prove

**Theorem 2 (Cauchy’s Integral Formula)** Let $f$ be holomorphic in an open set $\Omega$. Then

$$n(\gamma, z) f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta \quad (1)$$

where $\gamma \sim 0$ with respect to $\Omega$. 

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Proof: The prove is based on the following three claims.

Define \( g(z, \zeta) \) on \( \Omega \times \Omega \) by

\[
g(z, \zeta) = \begin{cases} \frac{f(\zeta) - f(z)}{\zeta - z} & \text{for } z \neq \zeta, \\ f'(z) & \text{for } z = \zeta. \end{cases}
\]

**Claim 1:** \( g \) is continuous on \( \Omega \times \Omega \) and holomorphic in each variable and \( g(z, \zeta) = g(\zeta, z) \).

Clearly \( g \) is continuous outside the diagonal in \( \Omega \times \Omega \). Let \((z_0, z_0)\) be a point on the diagonal and \( D \subset \Omega \) a disk with center \( z_0 \). Let \( z \neq \zeta \) in \( D \). Then by Theorem 8

\[
g(z, \zeta) - g(z_0, z_0) = f'(\zeta) + \frac{1}{2} f_2(z)(z - \zeta) - f'(z_0).
\]

So the continuous at \((z_0, z_0)\) is obvious.

For the holomorphy statement, it is clear that for each \( \zeta_0 \in \Omega \) the function

\[
z \mapsto g(z, \zeta_0)
\]

is holomorphic on \( \Omega - \zeta_0 \). Since

\[
\lim_{z \to \zeta_0} g(z, \zeta_0)(z - \zeta_0) = 0
\]

the point \( \zeta_0 \) is a removable singularity (Theorem 7, p.124), so

\[
z \mapsto g(z, \zeta_0)
\]

is indeed holomorphic on \( \Omega \). This proves Claim 1.
Let
\[ \Omega' = \{ z \in \mathbb{C} - (\gamma) : n(\gamma, z) = 0 \}. \]

Define function \( h \) on \( \mathbb{C} \) by
\[ h(z) = \frac{1}{2\pi i} \int_{\gamma} g(z, \zeta) \, d\zeta, \quad z \in \Omega; \] \hspace{1cm} (2)
\[ h(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta, \quad z \in \Omega'. \] \hspace{1cm} (3)

Since both expressions agree on \( \Omega \cap \Omega' \) and since \( \Omega \cup \Omega' = \mathbb{C} \), this is a valid definition.

**Claim 2:** \( h \) is holomorphic.

This is obvious on the open sets \( \Omega' \) and \( \Omega - \gamma \). To show holomorphy at \( z_0 \in \gamma \), consider a disk \( D \subset \Omega \) with center \( z_0 \). Let \( \delta \) be any closed curve in \( D \). Then
\[
\int_{\delta} h(z) \, dz = \frac{1}{2\pi i} \int_{\delta} \left( \int_{\gamma} g(z, \zeta) \, d\zeta \right) \, dz
= \frac{1}{2\pi i} \int_{\gamma} \left( \int_{\delta} g(z, \zeta) \, dz \right) \, d\zeta.
\]

For each \( \zeta \),
\[ z \mapsto g(z, \zeta) \]
is holomorphic on \( D \) (even \( \Omega \)). So by the Cauchy’s theorem for disks,
\[ \int_{\delta} g(z, \zeta) \, dz = 0. \]

Now the Morera’s Theorem implies \( h \) is holomorphic.

Now we can prove:

**Claim 3:** \( h \equiv 0 \), so (1) holds.

We have \( z \in \Omega' \) for \( |z| \) sufficiently large. So by (3),
\[ \lim_{z \to \infty} h(z) = 0. \]

By Liouville’s Theorem, \( h \equiv 0 \). \hspace{1cm} Q.E.D.
Proof of Theorem 1: To derive Cauchy’s theorem, let \( z_0 \in \Omega - \gamma \) and put

\[
F(z) = (z - z_0)f(z).
\]

By (1),

\[
\frac{1}{2\pi i} \int_{\gamma} f(z) \, dz = \frac{1}{2\pi i} \int_{\gamma} \frac{F(z)}{z - z_0} \, dz
\]

\[
= n(\gamma, z_0) F(z_0)
\]

\[
= 0.
\]

Q.E.D.

Note finally that Corollary 2 on p.142 is an immediately consequence of Cauchy’s Theorem.