Remarks on Lecture 15

In parts 4 and 5 (p. 154-160) some clarification of the use of the logarithm are called for.

Example 4 p.159

The relation

\[-z^{2\alpha} = e^{2\pi i \alpha} z^{2\alpha}\]

which is crucial for proof deserves explanation.

We consider the function

\[\log_{\theta} z = \log |z| + i \arg_{\theta} z\]

in the region \(\mathbb{C} - l_{\theta}\) (the plane with the ray \(l_{\theta}\) removed) where the angle is fixed by

\[\theta < \arg_{\theta} z < \theta + 2\pi.\]

In the problem of computing

\[\int_{0}^{\infty} x^{\alpha} R(x) \, dx\]

we consider

\[\log_{-\frac{\pi}{2}} (z)\]
in the plane $\mathbb{C}$ with the negative imaginary axis removed and use the Residue theorem on the contour in Fig. 4.13. As in the text we arrive at the integral
\[
\int_{-\infty}^{\infty} z^{2\alpha+1} R(z^2) \, dz = \int_{0}^{\infty} \left( z^{2\alpha+1} + (-z)^{2\alpha+1} \right) R(z^2) \, dz.
\]
On the right $z$ belongs to $(0, \infty)$ and
\[
\log_{-\frac{\pi}{2}}(-z) = \log |z| + \left( -\frac{\pi}{2} + \frac{\pi}{2} \right) i, \quad -\frac{\pi}{2} < \arg_{-\frac{\pi}{2}} z < \frac{3\pi}{2},
\]
\[
= \log_{-\frac{\pi}{2}}(z) + i\pi, \quad z > 0.
\]
Thus for $z > 0$,
\[
(-z)^{2\alpha+1} = e^{(2\alpha+1)\log_{-\frac{\pi}{2}}(-z)}
= e^{(2\alpha+1)(\log |z| + i\pi)}
= -e^{2\alpha i\pi z^{2\alpha+1}},
\]
so the last integrals combine to
\[
\left(1 - e^{2\alpha i\pi}\right) \int_{0}^{\infty} z^{2\alpha+1} R(z^2) \, dz.
\]
For $z > 0$ we have from the above
\[
\log_{-\frac{\pi}{2}}(z) = \log |z|,
\]
so
\[
\frac{1}{2\pi i} \left. \frac{1}{2\pi i} \int_{-\infty}^{\infty} z^{2\alpha+1} R(z^2) \, dz = \frac{1}{2\pi i} \left(1 - e^{2\alpha i\pi}\right) \int_{0}^{\infty} x^{2\alpha+1} R(x^2) \, dx \right|_{x=0} \right.

\int_{0}^{\infty} x^{2\alpha+1} R(x^2) \, dx.
\]
(1)
The left hand side of (1) is the sum of the residues of
\[
z^{2\alpha+1} R(z^2) = f(z)
\]
in the upper half plane. If
\[
R(z^2) = \frac{g(z)}{h(z)},
\]
where $g$ and $h$ are holomorphic, $g(a) \neq 0$, and $h$ has a simple zero at $a$, then
\[
\text{Res}_{z=a} f(z) = z^{2\alpha+1}(a) \frac{g(a)}{h'(a)}.
\]
(2)
Example: Exercise 3(g) p.161

To calculate
\[ \int_0^\infty x^{\frac{1}{3}} \frac{dx}{1 + x^2}, \]
we use \( x = t^2 \) and arrive at
\[ \int_{-\infty}^{\infty} z^{\frac{1}{3}} \frac{dz}{1 + z^4} \]
in (1). The poles in the upper half plane are
\[ z = e^{i\frac{\pi}{4}} \quad \text{and} \quad z = e^{i\left(\frac{\pi}{2} + \frac{\pi}{4}\right)}. \]

We use (2) to calculate the residues:

\[
\text{Res}_{z=e^{i\frac{\pi}{4}}} \left( z^{\frac{1}{3}} \frac{1}{1 + z^4} \right) = \frac{1}{4(e^{i\frac{\pi}{4}})^3} \left( e^{i\frac{\pi}{4}} \right)^{\frac{1}{3}} \log \left( e^{i\frac{\pi}{2}} \right) \left( e^{i\frac{\pi}{4}} \right)
\]
\[= 4(e^{i\frac{\pi}{4}})^{-\frac{1}{3}} \left( e^{i\frac{\pi}{4}} \right)^{\frac{1}{3}} \log \left( e^{i\frac{\pi}{2}} \right) \left( e^{i\frac{\pi}{4}} \right)
\]
\[= \frac{1}{4} e^{-i\frac{\pi}{3}}, \]
and

\[
\text{Res}_{z=e^{i\frac{3\pi}{4}}} \left( z^{\frac{1}{3}} \frac{1}{1 + z^4} \right) = \frac{1}{4(e^{i\frac{3\pi}{4}})^3} \left( e^{i\frac{3\pi}{4}} \right)^{\frac{1}{3}} \log \left( e^{i\left(-\frac{\pi}{4} +\frac{3\pi}{4}\right)} \right) \left( e^{i\frac{3\pi}{4}} \right)
\]
\[= \frac{1}{4} e^{-i\pi} \]

Thus (1) gives
\[ \frac{1}{4} e^{-i\frac{\pi}{2}} + \frac{1}{4} e^{-i\pi} = -\frac{1}{\pi} e^{\frac{1}{2} \pi i} \sin \frac{\pi}{3} \int_0^\infty x^{\frac{5}{3}} \frac{dx}{1 + x^2}, \]
so
\[ \int_0^\infty x^{\frac{5}{3}} \frac{dx}{1 + x^4} = \frac{\pi}{2\sqrt{3}}. \]
Example 5 p.160

The last four lines on the page are a bit misleading because the specific logarithm has already been chosen. So here is a completion of the proof after the equation

\[ \int_0^{\pi} \log(-2ie^{ix} \sin x) \, dx = 0. \]

We know (Lecture 2) that

\[ \log(z_1z_2) = \log z_1 + \log z_2, \quad \text{if} \quad -\pi < \text{Arg} z_1 + \text{Arg} z_2 < \pi. \quad (3) \]

Using this for \( z = 2 \sin x \) we get

\[ \int_0^{\pi} \log(2 \sin x) \, dx + \int_0^{\pi} \log(-ie^{ix}) \, dx = 0. \quad (4) \]

But

\[ \log(-i) = \frac{-\pi i}{2}, \quad \log e^{ix} = ix \quad (0 < x < \pi), \]

so since \(-\frac{\pi}{2} + x\) is in \((-\pi, \pi)\), (3) implies

\[ \log(-ie^{ix}) = -\frac{\pi i}{2} + ix. \]

Now (4) implies the result

\[ \int_0^{\pi} \log \sin \theta \, d\theta = -\pi \log 2. \]