Remarks on Lecture 6

Concerning Definition 13 p. 81, formula (11) shows that the definition does not depend on the choice of $z_1, z_2, z_3$.

Exercise 2 on page 88 requires a minor correction. For example $w = -z$ is hyperbolic according to definition on page 86, yet when written in the form

$$\frac{az + b}{cz + d}$$

with $ad - bc = 1$, we must take $a = -d = i$, so $a + d = 0$.

The transformation $w = z$ causes other ambiguities.

Thus we modify the definition a bit.

Definition 1

- $S$ is parabolic if either it is the identity or has exactly one fixed point.
- $S$ is strictly hyperbolic if $k > 0$ in (12) but $k \neq 1$.
- $S$ is elliptic if $|k| = 1$ in (12) p. 86 but $S$ is not identity.

Then the statement of Exercise 2 holds with hyperbolic replaced by strictly hyperbolic.
(i) the condition for exactly one fixed point for
\[ S_z = \frac{\alpha z + \beta}{\gamma z + \delta} \]
is
\[ (\alpha - \delta)^2 = -4\beta\gamma \] (wrong sign in text).
With the normalization \( \alpha\delta - \beta\gamma = 1 \) this amounts to
\[ (\alpha + \delta)^2 = 4 \] as desired.

(ii) Assume two fixed points are \( a \) and \( b \), so
\[ \frac{w - a}{w - b} = k \frac{z - a}{z - b}, \]
which we write as
\[ w = T z = \frac{\alpha z + \beta}{\gamma z + \delta}. \]
Put
\[ A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \]
and define
\[ \text{Tr}^2(T) = \frac{(\text{Trace} A)^2}{\det A}. \]
By linear algebra, \( \text{Trace}(BAB^{-1}) = \text{Trace}(A) \) and
\[ \det (BAB^{-1}) = \det A. \]
Define
\[ z_1 = S z = \frac{z - a}{z - b}, \]
\[ w_1 = S w = \frac{w - a}{w - b}. \]
Then
\[ \text{Tr}^2(T) = \text{Tr}^2(STS^{-1}). \]
Now
\[ w_1 = ST z = STS^{-1} z_1, \]
so
\[ w_1 = k z_1. \]

Then
\[ \text{Tr}^2(T) = \text{Tr}^2(STS^{-1}) = k + \frac{1}{k} + 2. \]

If \( T \) is strictly hyperbolic, we have
\[ k > 0, \quad k \neq 1, \]
so
\[ \text{Tr}^2(T) > 4, \]
which under the assumption
\[ \alpha \delta - \beta \gamma = 1 \]
amounts to
\[ (\alpha + \delta)^2 > 4 \]
as stated.

Conversely, if
\[ (\alpha + \delta)^2 > 4, \]
then \( k > 0 \). So the transformation
\[ w_1 = k z_1 \]
maps each line through 0 and \( \infty \) into itself. So \( T \) maps each circle \( C_1 \) into itself with \( k > 0 \). Thus \( T \) is strictly hyperbolic.

(iii) If \(|k| = 1\), then
\[ w_1 = e^{i \theta} z_1 \]
and we find
\[ \text{Tr}^2(T) = \left(2 \cos \frac{\theta}{2}\right)^2 < 4 \]
since the possibility \( \theta = 0 \) is excluded.

Conversely, if
\[ -2 < \alpha + \delta < 2, \quad \alpha \delta - \beta \gamma = 1 \]
we have
\[ \text{Tr}^2(T) = (\alpha + \delta)^2 = k + \frac{1}{k} + 2 < 4. \]
Writing
\[ k = re^{i \theta} \quad (r > 0) \]
this implies
\[
(r + \frac{1}{r}) \cos \theta + i \left( r - \frac{1}{r} \right) \sin \theta < 2,
\]
which implies
\[
r = 1 \quad \text{or} \quad \theta = 0 \quad \text{or} \quad \theta = \pi.
\]
If \( r = 1 \), then \( |k| = 1 \), so \( T \) is elliptic.
Since \( r + \frac{1}{r} \geq 2 \), the possibility \( \theta = 0 \) is ruled out.
Finally if \( \theta = \pi \), then \( k = -r \), so
\[
(\alpha + \delta)^2 = -r - \frac{1}{r} + 2.
\]
But \( r \geq 0 \), so since \( \alpha + \delta \) is real, this implies \( r = 1 \), so \( k = -1 \), and \( T \) is thus elliptic.