18.117 Assignment # 4

1. Let $\Delta \subseteq \mathbb{R}^d$ be a simple $m$-dimensional convex polytope, and let $\xi \in \mathbb{R}^d$. Assume that $\langle \xi, v' - v \rangle \neq 0$ for every pair of adjacent vertices $v$ and $v'$ of $\Delta$.

For $v \in \text{Vert}(\Delta)$, define

$$\text{ind}_\xi v = \# \{ v_i, \langle v_i - v, \xi \rangle < 0 \},$$

where $v_1, \ldots, v_m$ are the vertices adjacent to $v$.

Prove that

$$b_k \equiv \# \{ v \in \text{Vert}(\Delta), \text{ind}_\xi v = k \}$$

is independent of $\xi$.

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Hints for Problem 1:

(a) Let $F$ be a $k$-dimensional face of $\Delta$. Show that there is a unique vertex $v_F \in \text{Vert}(F)$ such that if $v_1, \ldots, v_k$ are the vertices of $F$ adjacent to $v_F$, then

$$\langle v_i - v_F, \xi \rangle > 0, \quad (\ast)$$

for all $i = 1, \ldots, k$.

(b) In particular, if $F = \Delta$, then there is one vertex $v_0$ with the property $(\ast)$. Conclude that $b_0 = 1$.

(c) Let $f_{m-1}$ be the number of $(m-1)$-dimensional faces of $\Delta$. Each such face $F$ has a unique vertex $v_F$ with property $(\ast)$. Show that there are $m$ such faces with $v_F = v_0$, and show that the number of such faces with $v_F \neq v_0$ is $b_1$. Conclude that $f_{m-1} = mb_0 + b_1$.

(d) Let $f_{m-2}$ be the number of $(m-2)$-dimensional faces of $\Delta$. Show that

$$f_{m-2} = \binom{m}{2} b_0 + \binom{m-1}{1} b_1 + b_2.$$

i. Sub-hint: The first summand counts the number of $(m-2)$-dimensional faces $F$ for which $v_F = v_0$.

The second summand counts the number of $(m-2)$-dimensional faces $F$ for which $v_F = v_{F'}$, where $F'$ is an $(m-1)$-dimensional face with $v_{F'} \neq v_0$.

The third summand counts all the other $(m-2)$-dimensional faces.

(e) In general, conclude that

$$f_{m-k} = \binom{m}{k} b_0 + \binom{m-1}{k-1} b_1 + \cdots + b_k = \sum_{\ell=0}^{k} \binom{m-\ell}{k-\ell} b_\ell.$$