Lecture 5

Notes about Exercise 1

**Lemma.** Let $U$ and $V$ be as in Theorem 1 above. $\beta \in \Omega^{0,q}(U)$, $\overline{\partial}\beta = 0$ then there exists $\alpha \in \Omega^{0,q-1}(U)$ such that $\overline{\partial}\alpha = \beta$ on $V$.

**Proof.** Choose a polydisk $W$ so that $\overline{W} \subset W, \overline{V} \subset U$. Choose $\rho \in C^\infty(W)$ with $\rho \equiv 1$ on a neighborhood of $V$. By theorem 1 there exists $\alpha_0 \in \Omega^{0,q-1}(W)$ so that $\overline{\partial}\alpha_0 = \beta$ on $W$. If we take

$$\alpha = \begin{cases} \rho \alpha_0 & \text{on } W \\ 0 & \text{on } U - W \end{cases}$$

then we have a solution.

We claim that the Dolbealt complex is exact on all degrees $q \geq 2$.

**Lemma.** Let $V_0, V_1, V_2, \ldots$ be a sequence of polydisks so that $\overline{V}_r \subset V_{r+1}$ and $\bigcup V_1 = U$ (exhaustion on $U$ by compact polydisk). There exists $\alpha_i \in \Omega^{0,q+1}(U)$ such that $\overline{\partial}\alpha_i = \beta$ on $V_r$ and such that $\alpha_{r+1} = \alpha_r$ on $V_{r-1}$.

**Proof.** By the previous lemma there exists $\alpha_r \in \Omega^{0,q-1}(U)$ with $\overline{\partial}\alpha_r = \beta$ on $V_r$. And for $\alpha_{r+1}, \alpha_r$ on $V_r$, $\overline{\partial}\alpha_{r+1} = \overline{\partial}\alpha_r = \beta$ on $V_r$, so $\overline{\partial}(\alpha_{r+1} - \alpha_r) = 0$ on $V_r$. Now $q \geq 2$ so we can find $\gamma \in \Omega^{0,q-1}(U)$ such that $\overline{\partial}\gamma = \alpha_{r+1} - \alpha_r$ on $V_{r-1}$. Then set $\alpha^{\text{new}}_r := \alpha^{\text{old}}_r + \overline{\partial}\gamma$. So $\overline{\partial}\alpha^{\text{new}}_r = \beta$ on $V_{r+1}$, $\alpha^{\text{new}}_{r+1} = \alpha_r$ on $V_{r-1}$. \qed
We get a global solution when we set \( \alpha = \alpha_r \) on \( V_{r-1} \) for all \( r \).

**(EXERCISE)** Prove exactness at \( q = 1 \), i.e. make this argument work for \( q = 1 \).

What does exactness mean for degree 1? Well

\[
\beta \in \Omega^{0,1}(U) \quad \beta = \sum f_id\bar{z}_i \quad f_i \in C^\infty(U)
\]

We need to show that there exists \( g \in \Omega^{0,0}(U) = C^\infty(U) \) so that \( \overline{\partial}g = \beta \), i.e.

\[
\frac{\partial g}{\partial \bar{z}_i} = f_i \quad i = 1, \ldots, n
\]

So the condition that \( \overline{\partial} \beta = 0 \) is just the integrability conditions.

So we have to show the following. That there exists a sequence of functions \( g_r \in C^\infty(U) \). \( V_0 \subset V_1 \subset \cdots \subset U \) such that \( \frac{\partial g}{\partial \bar{z}_i} = f_i \), \( i = 1, \ldots, n \) on \( V_r \) (easy consequence of lemma)

We can no longer say \( g_{r+1} = g_r \) on \( V_{r-1} \). But we can pick \( g_r \) such that \( |g_{r+1} - g_r| < \frac{1}{r^2} \) on \( V_{r-1} \).

**Hint** Choose \( g_r \in C^\infty(U) \) such that \( \frac{\partial g}{\partial \bar{z}_i} = f_i \) on \( V_r \). Look at \( g_{r+1} - g_r \) on \( V_r \). Note that \( \frac{\partial}{\partial \bar{z}_i}(g_{r+1} - g_r) = 0 \) on \( V_r \), so \( g_{r+1} - g_r \in O(V_r) \). On \( V_{r-1} \) we can expand by power series to get \( g_{r+1} - g_r = \sum a_n z^n \), and this series is actually uniformly convergent on \( V_{r-1} \). We try to modify \( g_{r+1}^{\text{old}} \) by setting \( g_{r+1}^{\text{new}} = g_{r+1}^{\text{old}} + P_N(z) \), where

\[
P_N(z) = \sum_{|\alpha| \leq N} a_\alpha z^\alpha
\]

(The exercise is due Feb 25th)

**More on Dolbeault Complex**

For polydisks the Dolbealt complex is acyclic (exact). But what about other kinds of open sets? The solution was obtained by Kohn in 1963.

Let \( U \) be open in \( \mathbb{C} \), \( \varphi : U \to \mathbb{R} \) be such that \( \varphi \in C^\infty(U) \).

**Definition.** \( \varphi \) is strictly pluri-subharmonic if for all \( p \in U \) the hermitian form

\[
a \in \mathbb{C}^n \mapsto \sum_{i,j} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(p) a_i \overline{a_j}
\]

is positive definite.

(This definition will be important later for Kaehler manifolds)

**Definition.** A \( C^\infty \) function \( \varphi : U \to \mathbb{R} \) is an exhaustion function if it is bounded from below and if for all \( c \in \mathbb{C} \)

\[
K_c = \{ p \in U | \varphi(p) \leq c \}
\]

is compact.

**Definition.** \( U \) is pseudoconvex if it possesses a strictly pluri-subharmonic exhaustion function.

**Examples**

1. \( U = \mathbb{C} \). If we take \( \varphi = |z|^2 = z\bar{z}, \frac{\partial \varphi}{\partial z \partial \overline{z}} = 1 \).
2. \( U = D \subset \mathbb{C} \)

\[
\varphi = \frac{1}{1 - |z|^2} \quad \frac{\partial \varphi}{\partial z \partial \overline{z}} = \frac{1 + |z|^2}{(1 - |z|^2)^2} > 0
\]

3. \( U \subset \mathbb{C}, U = D \setminus \{0\} = D^0 \), i.e. the punctured disk

\[
\varphi^o = \frac{1}{1 - |z|^2} + \log \frac{1}{|z|^2} \quad \frac{\partial \varphi^o}{\partial z \partial \overline{z}} = \frac{\partial \varphi}{\partial z \partial \overline{z}}
\]

because \( \log \) is harmonic. Note the extra term in \( \varphi^o \) is so the function will blow up at its point of discontinuity.
4. \( \mathbb{C}^n \ni U = D_1 \times \cdots \times D_n \), where \( D_i = |z_i|^2 < 1 \). Take
\[
\varphi = \sum \frac{1}{1 - |z_i|^2}
\]

5. \( \mathbb{C}^n \ni U, D_1^o \times \cdots \times D_k^o \times D_{k+1} \times \cdots \times D_n \)
\[
\varphi^o = \varphi + \sum_{i=1}^k \log \frac{1}{|z_i|^2}
\]

6. \( U \subseteq \mathbb{C}^n, U = B^n, |z|^2 = |z_1|^2 + \cdots + |z_n|^2 \).
\[
\varphi = \frac{1}{1 - |z|^2} \quad \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} = \frac{\delta_{ij}}{(1 + |z|^2)} + \frac{2z_i \bar{z}_j}{(1 - |z|^2)^3}
\]

**Theorem.** If \( U_i \subset \mathbb{C}^n, i = 1, 2 \) is pseudo-convex then \( U_1 \cap U_2 \) is pseudo-convex.

**Proof.** Take \( \varphi_i \) to be strictly pluri-subharmonic exhaustion functions for \( U_i \). Then set \( \varphi = \varphi_1 + \varphi_2 \) on \( U_1 \cap U_w \).

**Punchline:**

**Theorem.** The Dolbealt complex is exact on \( U \) if and only if \( U \) is pseudo-convex.

This takes 150 pages to prove, so we’ll just take it as fact.

The Dolbealt complex is the left side of the bi-graded de Rham complex.

There is another interesting complex. For example if we let \( A^0 = \ker \bar{\partial} : \Omega^{p,0} \rightarrow \Omega^{p,1}, \bar{\partial} \bar{\partial} + \partial \partial = 0 \) and \( \omega \in A^r \) then \( \partial \omega \in A^{r+1} \) and we get a complex
\[
A^0 \xrightarrow{\partial} A^1 \xrightarrow{\partial} A^2 \xrightarrow{\partial} \cdots
\]