4.5 Pseudodifferential operators on $T^n$

In this section we will prove Theorem 4.2 for elliptic operators on $T^n$. Here’s a road map to help you navigate this section. §4.5.1 is a succinct summary of the material in §4. Sections 4.5.2, 4.5.3 and 4.5.4 are a brief account of the theory of pseudodifferential operators on $T^n$ and the symbolic calculus that’s involved in this theory. In §4.5.5 and 4.5.6 we prove that an elliptic operator on $T^n$ is right invertible modulo smoothing operators (and that its inverse is a pseudodifferential operator). Finally, in §4.5.7, we prove that pseudodifferential operators have a property called “pseudolocality” which makes them behave in some ways like differential operators (and which will enable us to extend the results of this section from $T^n$ to arbitrary compact manifolds).

Some notation which will be useful below: for $a \in \mathbb{R}^n$ let

$$\langle a \rangle = (|a|^2 + 1)^{1/2}.$$ 

Thus

$$|a| \leq \langle a \rangle$$

and for $|a| \geq 1$

$$\langle a \rangle \leq 2|a|.$$

4.5.1 The Fourier inversion formula

Given $f \in \mathcal{C}^\infty(T^n)$, let $c_k(f) = \langle f, e^{ikx} \rangle$. Then:

1) $c_k(D^{\alpha}f) = k^\alpha c_k(f)$.

2) $|c_k(f)| \leq C_r \langle k \rangle^{-r}$ for all $r$.

3) $\sum c_k(f)e^{ikx} = f$. 

Lecture 18
Let $S$ be the space of functions, $g : \mathbb{Z}^n \to \mathbb{C}$ satisfying

$$|g(k)| \leq C_r(k)^{-r}$$

for all $r$. Then the map

$$F : C^\infty(T^n) \to S, \quad Ff(k) = c_k(f)$$

is bijective and its inverse is the map,

$$g \in S \to \sum g(k) e^{ikx}.$$

### 4.5.2 Symbols

A function $a : T^n \times \mathbb{R}^n \to \mathbb{C}$ is an $S^m$ if, for all multi-indices, $\alpha$ and $\beta$,

$$|D_\alpha^\beta a| \leq C_{\alpha,\beta} |\xi|^{m-|\beta|}.$$  \hspace{1cm} (5.2.1)

**Examples**

1) $a(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha$, $a_\alpha \in C^\infty(T^n)$.
2) $(\xi)^m$.
3) $a \in S^\ell$ and $b \in S^m \Rightarrow ab \in S^{\ell+m}$.
4) $a \in S^m \Rightarrow D_\alpha^\beta a \in S^{m-|\beta|}$.

### The asymptotic summation theorem

Given $b_i \in S^{m-i}$, $i = 0, 1, \ldots$, there exists a $b \in S^m$ such that

$$b - \sum_{j<i} b_j \in S^{m-i}. \hspace{1cm} (5.2.2)$$

**Proof. Step 1.** Let $\ell = m + \epsilon, \epsilon > 0$. Then

$$|b_i(x, \xi)| < C_{i} |\xi|^{m-i} = \frac{C_{i} |\xi|^{\ell-i}}{|\xi|^{\ell}}.$$  \hspace{1cm} (5.2.3)

Thus, for some $\lambda_i$,

$$|b_i(x, \xi)| < \frac{1}{2^i} |\xi|^{\ell-i}$$

for $|\xi| > \lambda_i$. We can assume that $\lambda_i \to +\infty$ as $i \to +\infty$. Let $\rho \in C^\infty(\mathbb{R})$ be bounded between 0 and 1 and satisfy $\rho(t) = 0$ for $t < 1$ and $\rho(t) = 1$ for $t > 2$. Let

$$b = \sum \rho \left( \frac{|\xi|}{\lambda_i} \right) b_i(x, \xi). \hspace{1cm} (5.2.3)$$

Then $b$ is in $C^\infty(T^n \times \mathbb{R}^n)$ since, on any compact subset, only a finite number of summands are non-zero. Moreover, $b - \sum_{j<i} b_j$ is equal to:

$$\sum_{j<i} \left( \rho \left( \frac{|\xi|}{\lambda_j} \right) - 1 \right) b_j + b_i + \sum_{j>i} \rho \left( \frac{|\xi|}{\lambda_j} \right) b_j.$$  \hspace{1cm} (5.2.3)

The first summand is compactly supported, the second summand is in $S^{m-1}$ and the third summand is bounded from above by

$$\sum_{k>i} \frac{1}{2^k} |\xi|^{\ell-k}.$$
which is less than $(\xi)^{\ell-(i+1)}$ and hence, for $\epsilon < 1$, less than $(\xi)^{m-i}$.

**Step 2.** For $|\alpha| + |\beta| \leq N$ choose $\lambda_i$ so that

$$|D_\xi^\beta D_\xi^\alpha b_i(x, \xi)| \leq \frac{1}{2^i}(\xi)^{\ell-i-|\beta|}$$

for $\lambda_i < |\xi|$. Then the same argument as above implies that

$$D_\xi^\alpha D_\xi^\beta (b - \sum b_j) \leq C_N(\xi)^{m-i-|\beta|}$$

(5.2.4)

for $|\alpha| + |\beta| \leq N$.

**Step 3.** The sequence of $\lambda_i$’s in step 2 depends on $N$. To indicate this dependence let’s denote this sequence by $\lambda_{i,N}$, $i = 0, 1, \ldots$. We can, by induction, assume that for all $i$, $\lambda_{i,N} \leq \lambda_{i,N+1}$. Now apply the Cantor diagonal process to this collection of sequences, i.e., let $\lambda_i = \lambda_{i,i}$. Then $b$ has the property (5.2.4) for all $N$.

We will denote the fact that $b$ has the property (5.2.2) by writing

$$b \sim \sum b_i.$$  

(5.2.5)

The symbol, $b$, is not unique, however, if $b \sim \sum b_i$ and $b' \sim \sum b_i$, $b - b'$ is in the intersection, $\bigcap S^\ell$, $-\infty < \ell < \infty$.

### 4.5.3 Pseudodifferential operators

Given $a \in \mathcal{S}^m$ let

$$T_a^0 : S \rightarrow \mathcal{C}^\infty(T^n)$$

be the operator

$$T_a^0 g = \sum a(x,k)g(k)e^{ikx}. $$

Since

$$|D^\alpha a(x,k)e^{ikx}| \leq C_\alpha \langle k \rangle^{m+|\alpha|}$$

and

$$|g(k)| \leq C_\alpha \langle k \rangle^{-(m+n+|\alpha|+1)}$$

this operator is well-defined, i.e., the right hand side is in $\mathcal{C}^\infty(T^n)$. Composing $T_a^0$ with $F$ we get an operator

$$T_a : \mathcal{C}^\infty(T^n) \rightarrow \mathcal{C}^\infty(T^n).$$

We call $T_a$ the pseudodifferential operator with symbol $a$.

Note that

$$T_a e^{ikx} = a(x,k)e^{ikx}. $$

Also note that if

$$P = \sum_{|\alpha| \leq m} a_\alpha(x)D^\alpha$$

(5.3.1)

and

$$p(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x)\xi^\alpha. $$

(5.3.2)

Then

$$P = T_p.$$
4.5.4 The composition formula

Let \( P \) be the differential operator (5.3.1). If \( a \) is in \( S' \) we will show that \( PT_a \) is a pseudodifferential operator of order \( m + r \). In fact we will show that

\[
PT_a = T_{poa}
\]

where

\[
p \circ a(x, \xi) = \sum_{|\alpha| \leq m} \frac{1}{\beta!} \partial_\xi^\beta p(x, \xi) D_x^\beta a(x, \xi)
\]

and \( p(x, \xi) \) is the function (5.3.2).

Proof. By definition

\[
PT_a e^{ikx} = Pa(x, k)e^{ikx} = e^{ikx} e^{-i k x} Pe^{i k x} a(x, k).
\]

Thus \( PT_a \) is the pseudodifferential operator with symbol

\[
e^{-i x \xi} Pe^{i x \xi} a(x, \xi).
\]

However, by (5.3.1):

\[
e^{-i x \xi} Pe^{i x \xi} u(x) = \sum a_\alpha(x) e^{-i x \xi} D^\alpha e^{i x \xi} u(x)
= \sum a_\alpha(x) (D + \xi)^\alpha u(x)
= P(x, D + \xi) u(x).
\]

Moreover,

\[
p(x, \eta + \xi) = \sum \frac{1}{\beta!} \frac{\partial}{\partial \xi^\beta} p(x, \xi) \eta^\beta,
\]

so

\[
p(x, D + \xi) u(x) = \sum \frac{1}{\beta!} \frac{\partial}{\partial \xi^\beta} p(x, \xi) D^\beta u(x)
\]

and if we plug in \( a(x, \xi) \) for \( u(x) \) we get, by (5.4.3), the formula (5.4.2) for the symbol of \( PT_a \).

4.5.5 The inversion formula

Suppose now that the operator (5.3.1) is elliptic. We will prove below the following inversion theorem.

**Theorem.** There exists an \( a \in S^{-m} \) and an \( r \in \bigcap S^\ell, -\infty < \ell < \infty \), such that

\( PT_a = I - T_r \).

Proof. Let

\[
p_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha.
\]

By ellipticity \( p_m(x, \xi) \neq 0 \) for \( \xi \neq 0 \). Let \( \rho \in C^\infty(\mathbb{R}) \) be a function satisfying \( \rho(t) = 0 \) for \( t < 1 \) and \( \rho(t) = 1 \) for \( t > 2 \). Then the function

\[
a_0(x, \xi) = \rho(|\xi|) \frac{1}{p_m(x, \xi)}
\]

is well-defined and belongs to \( S^{-m} \). To prove the theorem we must prove that there exist symbols \( a \in S^{-m} \) and \( r \in \bigcap S^\ell, -\infty < \ell < \infty \), such that

\( p \circ q = 1 - r \).

We will deduce this from the following two lemmas.
Lemma. If $b \in S^i$ then $b - p \circ a_0 b$ is in $S^{i-1}$.

Proof. Let $q = p - p_m$. Then $q \in S^{m-1}$ so $q \circ a_0 b$ is in $S^{i-1}$ and by (5.4.2)

$$p \circ a_0 b = p_m \circ a_0 b + q \circ a_0 b = p_m a_0 b + \cdots = b + \cdots$$

where the dots are terms of order $i - 1$.

Lemma. There exists a sequence of symbols $a_i \in S^{-m-i}$, $i = 0, 1, \ldots$, and a sequence of symbols $r_i \in S^{-i}$, $i = 0, \ldots$, such that $a_0$ is the symbol (5.5.1), $r_0 = 1$ and

$$p \circ a_i = r_i - r_{i+1}$$

for all $i$.

Proof. Given $a_0, \ldots, a_{i-1}$ and $r_0, \ldots, r_i$, let $a_i = r_i a_0$ and $r_{i+1} = r_i - p \circ a_i$. By Lemma 4.5.5, $r_{i+1} \in S^{-i-1}$.

Now let $a \in S^{-m}$ be the “asymptotic sum” of the $a_i$’s

$$a \sim \sum a_i.$$ 

Then

$$p \circ a \sim \sum p \circ a_i = \sum_{i=1}^{\infty} r_i - r_{i+1} = r_0 = 1,$$

so $1 - p \circ a \sim 0$, i.e., $r = 1 - p \circ q$ is in $\bigcap S^{\ell}$, $-\infty < \ell < \infty$.

4.5.6 Smoothing properties of $\Psi DO$’s

Let $a \in S^\ell$, $\ell < -m - n$. We will prove in this section that the sum

$$K_a(x, y) = \sum a(x, k) e^{i(k-x,y)}$$

is in $C^m(T^\beta \times T^n)$ and that $T_a$ is the integral operator associated with $K_a$, i.e.,

$$T_a u(x) = \int K_a(x, y) u(y) \, dy.$$

Proof. For $|\alpha| + |\beta| \leq m$

$$D^\alpha_x D^\beta_y a(x, k) e^{i(k-x,y)}$$

is bounded by $(k)^{\ell + |\alpha| + |\beta|}$ and hence by $(k)^{\ell + m}$. But $\ell + m < -n$, so the sum

$$\sum D^\alpha_x D^\beta_y a(x, k) e^{i(k-x,y)}$$

converges absolutely. Now notice that

$$\int K_a(x, y) e^{iky} \, dy = a(x, k) e^{ikx} = T_a e^{ikx}.$$

Hence $T_a$ is the integral operators defined by $K_a$. Let

$$S^{-\infty} = \bigcap S^{\ell}, \quad -\infty < \ell \infty.$$ 

If $a$ is in $S^{-\infty}$, then by (5.6.1), $T_a$ is a smoothing operator.
4.5.7 Pseudolocality

We will prove in this section that if \( f \) and \( g \) are \( C^\infty \) functions on \( T^n \) with non-overlapping supports and \( a \) is in \( \mathcal{S}^m \), then the operator

\[
    u \in C^\infty(T^n) \to fT_ag \quad (5.7.1)
\]

is a smoothing operator. (This property of pseudodifferential operators is called pseudolocality.) We will first prove:

**Lemma.** If \( a(x,\xi) \) is in \( \mathcal{S}^m \) and \( w \in \mathbb{R}^n \), the function,

\[
    a_w(x,\xi) = a(x,\xi + w) - a(x,\xi) \quad (5.7.2)
\]

is in \( \mathcal{S}^{m-1} \).

**Proof.** Recall that \( a \in \mathcal{S}^m \) if and only if

\[
    |D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta} |\xi|^{-|\beta|}. 
\]

From this estimate is is clear that if \( a \) is in \( \mathcal{S}^m \), \( a(x,\xi + w) \) is in \( \mathcal{S}^m \) and \( \partial a/\partial \xi_i(x,\xi) \) is in \( \mathcal{S}^{m-1} \), and hence that the integral

\[
    a_w(x,\xi) = \int_0^1 \sum_i \frac{\partial a}{\partial \xi_i}(x,\xi + tw) \, dt
\]

in \( \mathcal{S}^{m-1} \).

Now let \( \ell \) be a large positive integer and let \( a \) be in \( \mathcal{S}^m \), \( m < -n - \ell \). Then

\[
    K_a(x,y) = \sum a(x,k)e^{ik(x-y)}
\]

is in \( C^\ell(T^n \times T^n) \), and \( T_a \) is the integral operator defined by \( K_a \). Now notice that for \( w \in \mathbb{Z}^n \)

\[
    (e^{-i(x-y)w} - 1)K_a(x,y) = \sum a_w(x,k)e^{ik(x-y)}, \quad (5.7.3)
\]

so by the lemma the left hand side of (5.7.3) is in \( C^{\ell+1}(T^n \times T^n) \). More generally,

\[
    (e^{-i(x-y)w} - 1)^N K_a(x,y) \quad (5.7.4)
\]

is in \( C^{\ell+N}(T^n \times T^n) \). In particular, if \( x \neq y \), then for some \( 1 \leq i \leq n \), \( x_i - y_i \neq 0 \mod 2\pi \mathbb{Z} \), so if

\[
    w = (0, 0, \ldots, 1, 0, \ldots, 0),
\]

(\( a \) “1” in the \( i \)-th slot), \( e^{i(x-y)w} \neq 1 \) and, by (5.7.4), \( K_a(x,y) \) is \( C^{\ell+N} \) a neighborhood of \( (x,y) \). Since \( N \) can be arbitrarily large we conclude

**Lemma.** \( K_a(x,y) \) is a \( C^\infty \) function on the complement of the diagonal in \( T^n \times T^n \).

Thus if \( f \) and \( g \) are \( C^\infty \) functions with non-overlapping support, \( fT_ag \) is the smoothing operator, \( T_K \), where

\[
    K(x,y) = f(x)K_a(x,y)g(y). \quad (5.7.5)
\]

We have proved that \( T_a \) is pseudolocal if \( a \in \mathcal{S}^m \), \( m < -n - \ell \), \( \ell \) a large positive integer. To get rid of this assumption let \( \langle D \rangle^N \) be the operator with symbol \( \langle \xi \rangle^N \). If \( N \) is an even positive integer

\[
    \langle D \rangle^N = (\sum D_i^2 + I)^{N/2}
\]

is a differential operator and hence is a local operator: if \( f \) and \( g \) have non-overlapping supports, \( f\langle D \rangle^Ng \) is identically zero. Now let \( a_N(x,\xi) = a(x,\xi)\langle \xi \rangle^{-N} \). Since \( a_N \in \mathcal{S}^{m-N} \), \( T_{a_N} \) is pseudolocal for \( N \) large. But \( T_a = T_{a_N} \langle D \rangle^N \), so \( T_a \) is the composition of an operator which is pseudolocal with an operator which is local, and therefore \( T_a \) itself is pseudolocal.
4.6 Elliptic operators on open subsets of $T^n$

Let $U$ be an open subset of $T^n$. We will denote by $\iota_U : U \to T^n$ the inclusion map and by $\iota_U^* : C^\infty(T^n) \to C^\infty(U)$ the restriction map: let $V$ be an open subset of $T^n$ containing $U$ and

$$P = \sum_{|\alpha| \leq m} a_\alpha(x)D^\alpha, \quad a_\alpha(x) \in C^\infty(V)$$

an elliptic $m^{th}$ order differential operator. Let

$$P^t = \sum_{|\alpha| \leq m} D^\alpha\pi_\alpha(x)$$

be the transpose operator and

$$P_m(x,\xi) = \sum_{|\alpha| = m} a_\alpha(x)\xi^\alpha$$

the symbol at $P$. We will prove below the following localized version of the inversion formula of § 4.5.5.

**Theorem.** There exist symbols, $a \in S^{-m}$ and $r \in S^{-\infty}$ such that

$$\iota_U^*P\iota_U^*(I - T_r) = (4.6.1)$$

**Proof.** Let $\gamma \in C^\infty_0(V)$ be a function which is bounded between 0 and 1 and is identically 1 in a neighborhood of $U$. Let

$$Q = PP^t\gamma + (1 - \gamma)(\sum D^2)$$

This is a globally defined $2m^{th}$ order differential operator in $T^n$ with symbol,

$$\gamma(x)|P_m(x,\xi)|^2 + (1 - \gamma(x))|\xi|^{2m}$$

and since (4.6.2) is non-vanishing on $T^n \times (R^n - 0)$, this operator is elliptic. Hence, by Theorem 4.5.5, there exist symbols $b \in S^{-2m}$ and $r \in S^{-\infty}$ such that

$$QTb = I - T_r.$$ 

Let $T_a = P^t\gamma T_b$. Then since $\gamma \equiv 1$ on a neighborhood of $U$,

$$\iota_U^*(I - T_r) = \iota_U^*QTb$$

$$= \iota_U^*(PP^t\gamma T_b + (1 - \gamma)(\sum D^2)T_b)$$

$$= \iota_U^*P^t\gamma T_b$$

$$= P\iota_U^*P^t\gamma T_b = P\iota_U^*T_a.$$ 

4.7 Elliptic operators on compact manifolds

Let $X$ be a compact $n$ dimensional manifold and

$$P : C^\infty(X) \to C^\infty(X)$$

an elliptic $m^{th}$ order differential operator. We will show in this section how to construct a parametrix for $P$: an operator

$$Q : C^\infty(X) \to C^\infty(X)$$

such that $I - PQ$ is smoothing.

Let $V_i, i = 1, \ldots, N$ be a covering of $X$ by coordinate patches and let $U_i, i = 1, \ldots, N, U_i \subset V_i$ be an open covering which refines this covering. We can, without loss of generality, assume that $V_i$ is an open subset of the hypercube

$$\{x \in R^n \quad 0 < x_i < 2\pi \quad i = 1, \ldots, n\}$$
and hence an open subset of $T^n$. Let 
\[ \{ \rho_i \in C_0^\infty(U_i), \ i = 1, \ldots, N \} \]
be a partition of unity and let $\gamma_i \in C_0^\infty(U_i)$ be a function which is identically one on a neighborhood of the support of $\rho_i$. By Theorem 4.6, there exist symbols $a_i \in S^{-m}$ and $r_i \in S^{-\infty}$ such that on $T^n$:
\[ P_{i_U}^* T a_i = \iota_{i_U}^* (I - T r_i). \] (4.7.1)
Moreover, by pseudolocality $(1 - \gamma_i) T a_i \rho_i$ is smoothing, so
\[ \gamma_i T a_i, \rho_i = \iota_{i_U}^* T a_i \rho_i \]
and
\[ P \gamma_i T a_i, \rho_i - P \iota_{i_U}^* T a_i, \rho_i \]
are smoothing. But by (4.7.1)
\[ P \iota_{i_U}^* T a_i, \rho_i - \rho_i I \]
is smoothing. Hence
\[ P \gamma_i T a_i, \rho_i - \rho_i I \] (4.7.2)
is smoothing as an operator on $T^n$. However, $P \gamma_i T a_i, \rho_i$ and $\rho_i I$ are globally defined as operators on $X$ and hence (4.7.2) is a globally defined smoothing operator. Now let $Q = \sum \gamma_i T a_i, \rho_i$ and note that by (4.7.2)
\[ PQ - I \]
is a smoothing operator.

This concludes the proof of Theorem 4.3, and hence, modulo proving Theorem 4.3. This concludes the proof of our main result: Theorem 4.2. The proof of Theorem 4.3 will be outlined, as a series of exercises, in the next section.

### 4.8 The Fredholm theorem for smoothing operators

Let $X$ be a compact $n$-dimensional manifold equipped with a smooth non-vanishing measure, $dx$. Given $K \in C^\infty(X \times X)$ let
\[ T_K : C^\infty(X) \to C^\infty(X) \]
be the smoothing operator 3.1.

**Exercise 1.** Let $V$ be the volume of $X$ (i.e., the integral of the constant function, 1, over $X$). Show that if
\[ \max |K(x, y)| < \frac{\epsilon}{V}, \quad 0 < \epsilon < 1 \]
then $I - T_K$ is invertible and its inverse is of the form, $I - T_L, L \in C^\infty(X \times X)$.

**Hint 1.** Let $K_i = K \circ \cdots \circ K$ ($i$ products). Show that $\sup |K_i(x, y)| < C\epsilon^i$ and conclude that the series
\[ \sum K_i(x, y) \] (4.8.1)
converges uniformly.

**Hint 2.** Let $U$ and $V$ be coordinate patches on $X$. Show that on $U \times V$
\[ D^\alpha_x D^\beta_y K_i(x, y) = K_\alpha \circ K_{i-2} \circ K_\beta(x, y) \]
where $K_\alpha(x, z) = D^\alpha_x K(x, z)$ and $K_\beta(z, y) = D^\beta_y K(z, y)$. Conclude that not only does (8.1) converge on $U \times V$ but so do its partial derivatives of all orders with respect to $x$ and $y$.

**Exercise 2.** (finite rank operators.) $T_K$ is a finite rank smoothing operator if $K$ is of the form:
\[ K(x, y) = \sum_{i=1}^N f_i(x) g_i(y). \] (4.8.2)
(a) Show that if $T_K$ is a finite rank smoothing operator and $T_L$ is any smoothing operator, $T_K T_L$ and $T_L T_K$ are finite rank smoothing operators.

(b) Show that if $T_K$ is a finite rank smoothing operator, the operator, $I - T_K$, has finite dimensional kernel and co-kernel.

**Hint.** Show that if $f$ is in the kernel of this operator, it is in the linear span of the $f_i$'s and that $f$ is in the image of this operator if

$$\int f(y)g_i(y)\,dy = 0, \quad i = 1, \ldots, N.$$ 

**Exercise 3.** Show that for every $K \in C^\infty(X \times X)$ and every $\epsilon > 0$ there exists a function, $K_1 \in C^\infty(X \times X)$ of the form (4.8.2) such that

$$\sup |K - K_1|(x, y) < \epsilon.$$ 

**Hint.** Let $\mathcal{A}$ be the set of all functions of the form (4.8.2). Show that $\mathcal{A}$ is a subalgebra of $C(X \times X)$ and that this subalgebra separates points. Now apply the Stone–Weierstrass theorem to conclude that $\mathcal{A}$ is dense in $C(X \times X)$.

**Exercise 4.** Prove that if $T_K$ is a smoothing operator the operator

$$I - T_K : C^\infty(X) \to C^\infty(X)$$

has finite dimensional kernel and co-kernel.

**Hint.** Show that $K = K_1 + K_2$ where $K_1$ is of the form (4.8.2) and $K_2$ satisfies the hypotheses of exercise 1. Let $I - T_L$ be the inverse of $I - T_{K_2}$. Show that the operators

$$(I - T_K) \circ (I - T_L)$$

$$(I - T_L) \circ (I - T_K)$$

are both of the form: identity minus a finite rank smoothing operator. Conclude that $I - T_K$ has finite dimensional kernel and co-kernel.

**Exercise 5.** Prove Theorem 4.3.