Lecture 23

For the next few days we’re assuming that $B$ is symplectic and $V = V^{2n}$. Choose a Darboux basis $e_1, f_1, \ldots, e_n, f_n$. Check that $L_B : V \to V^*$ is the map

$$\{e_i \to -f_i^*, f_i \to e_i^*\}$$

where $e_i^*, f_i^*$ are the dual vectors. In the symplectic case $B^\sharp = -B$ and $L_B^\sharp = -L$.

Say that $\omega \in \Lambda^2 V,$

$$\omega = \sum e_i \wedge f_i$$

Then we have the operation $L : \Lambda^p \to \Lambda^{p+2}$, given by $\alpha \mapsto \omega \wedge \alpha$ and also its transpose $L^t : \Lambda^{p+2} \to \Lambda^p$.

Theorem (Kaehler, Weil). $[L, L^t] = (p - n) \text{Id}$

Proof. $L = \sum_i L_{e_i} L_{f_i}$, so

$$L^t = \sum_i L_{e_i}^t L_{f_i}^t = \sum_i \iota_{f_i^*} \iota_{e_i^*}$$

It’s easy to see that Kaehler-Weil holds when $n = 2$.

For $n$-dimensions

$$L = \sum_i L_i \quad L_i = L_{e_i} L_{f_i} \quad L^t = \sum_i L_i^t \quad L_i^t = \iota_{f_i^*} \iota_{e_i^*}$$

$V_i = \text{span}\{e_i, f_i\}$, then $\Lambda^p = \text{span}\beta_1 \wedge \cdots \wedge \beta_n$ where $\beta_i \in \Lambda^p(V_i)$.

Note that

$$L_i \beta_1 \wedge \cdots \wedge \beta_n = \beta_1 \wedge \cdots \wedge (L_i \beta_i) \wedge \cdots \wedge \beta_n$$

and

$$L_i^t (\beta_1 \wedge \cdots \wedge \beta_n) = \beta_1 \wedge \cdots \wedge (L_i^t \beta_i) \wedge \cdots \wedge \beta_n$$

If $i \neq j$, then $L_i L_j^t = L_j^t L_i$. So

$$[L, L^t] \beta_1 \wedge \cdots \wedge \beta_n = \sum_i \beta_1 \wedge \cdots \wedge [L_i, L_i^t] \beta_i \wedge \cdots \wedge \beta_n$$

$$= \sum_{i,j} (p_i - 1) \beta_1 \wedge \cdots \wedge \beta_n = (p - n) \beta_1 \wedge \cdots \wedge \beta_n$$

$\square$