Lecture 24

Proposition. \( L^t = *^{-1} L* \)

Proposition. \( u \in V \) then \([L^t_u, L] = -L_u\).

Proof. Proof omitted.

Let \((X^{2n}, \omega)\) be a compact symplectic manifold. Let \(x \in X\) and \(V = T^*_x\). Notice

(a) From \(\omega_x\) we get a symplectic bilinear form on \(T_x\).

(b) From this form we get an identification \(T_x \to T^*_x\).

(c) Hence from 1, 2 we get a symplectic bilinear from \(B_x\) on \(V\).

(d) From \(B_x\) we get a \(*\)-operator

\[ *_x : \Lambda^p(T^*_x) \to \Lambda^{2n-p}(T^*_x) \]

(e) This gives us a \(*\)-operator on forms

\[ * : \Omega^p(X) \to \Omega^{2n-p}(X) \]

We can define a symplectic version of the \(L^2\) inner product on \(\Omega^p\) as follows. Take \(\alpha, \beta \in \Omega^p\) and define

\[ \langle \alpha, \beta \rangle = \int_X \alpha \wedge *\beta \]

(Note: This is not positive definite or anything, its just a pairing)

Take \(\alpha \in \Omega^{p-1}, \beta \in \Omega^{p}\). Then look at

\[ d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-1)^{p-1} \alpha \wedge d*\beta \]

\[ = d\alpha \wedge *\beta + (-1)^{p-1} \alpha \wedge *(d\alpha \wedge *\beta) \]

Since \(\int_X d(\alpha \wedge *\beta) = 0\), we integrate both sides of the above and get

\[ \int_X \alpha \wedge *\beta = (-1)^p \int \alpha \wedge *(d\alpha \wedge *\beta) \]

If we introduce the notation \(\delta = (-1)^p *^{-1} d*\) on \(\Omega^p\) then

\[ \langle d\alpha, \beta \rangle = \langle \alpha, \delta \beta \rangle \]

Now, given the mapping \(L : \Omega^p \to \Omega^{p+2}\), \(L\alpha = \omega \wedge \alpha\) we have the following theorem

**Theorem.** \([\delta, L] = d\).

This identity has no analogue in ordinary Hodge Theory. This is very important.

**Proof.** \(x \in X, \xi \in T^*_x\), then \(\sigma(d)(x, \xi) = iL\xi\). On \(\Lambda^p\), \(\delta = (-1)^p *^{-1} d*\), so \(\sigma(d)(x, \xi) = (-1)^p i*^{-1} L\xi = -iL\xi\).

Then

\[ \sigma([\delta, L]) = i[L\xi, L] = iL\xi = \sigma(d)(x, \xi) \]

so \([\delta, L]\) and \(d\) have the same symbol.

Now, \(d[\delta, L]\) are first order DO’s mapping \(\Omega^p \to \Omega^{p+1}\), so \(d - [\delta, L] : \Omega^p \to \Omega^{p+1}\) is a first order DO. We want to show that this is 0.

Let \((U, x_1, \ldots, x_n, y_1, \ldots, y_n)\) be a Darboux coordinate patch. Consider \(u = \beta_1 \wedge \cdots \wedge \beta_n\) where \(\beta_i = 1, dx_i, dy_i\) or \(dx_i \wedge dy_i\).

These de Rham forms are a basis at each point of \(\Lambda(T^*_x)\).

\(Lu = \omega \wedge u\) is again a form of this type since \(\omega = \sum dx_i \wedge dy_i\) is of this form. Also \(*u\) is of this form.

Note that \(d = 0\) on a form of this type, hence \(\delta = *^{-1} d*\) is 0 on a form of this type. Thus \([\delta, L] - d\) is 0 on a form of this type.