Lecture 25

Symplectic Hodge Theory

\( (X^{2n}, \omega) \) be a compact symplectic manifold. From \( x \in X \) we get \( \omega_x \to B_x \) a non-degenerate bilinear form on \( T_x^* \), and so induces a non-degenerate bilinear from on \( \Lambda^p(T_x^*) \).

Define \( \langle \cdot, \cdot \rangle_{L^2} \) on \( \Omega^p \) as follows. Take \( \Omega = \omega^n/n! \), a symplectic volume form, \( \alpha, \beta \in \Omega^p \)

\[
\langle \alpha, \beta \rangle = \int_X B_x(\alpha, \beta) \Omega = \int_X \alpha \wedge \beta
\]

Remarks:

(a) In symplectic geometry \( *^2 = id \), \( * = *^{-1} \).
(b) \( \langle \cdot, \cdot \rangle \) is anti-symmetric on \( \Omega^p \), \( p \) odd and symmetric on \( \Omega^p \), \( p \) even.
(c) \( [L^i, \delta^i] = d^i = \delta \). And \( \delta^i = (d^i)^t = -d \), so \( [d, L^i] = \delta \).

Consider the Laplace operator \( \delta \delta = \delta d + \delta d \). Now, in the symplectic world, \( \Delta = 0 \). We'll prove this: \( \delta = [d, L^i] = dL^i - L^i d, \) so \( \delta \delta = -dL^i d \) and \( \delta d = dL^i, \) so \( \Delta = 0 \).

So for symplectic geometry we work with the bicomplex \( (\Omega, d, \delta) \). We're going to use symplectic geometry to prove the Hard Lefshetz theorem for Kaehler manifolds.

Let \( (X^{2n}, \omega) \) be a compact Kaehler manifold. Then we have the following operation in cohomology

\[
\gamma : H^p(X, \mathbb{C}) \to H^{p+2}(X) \quad c \mapsto [\omega] \sim c
\]

Theorem (Hard Lefshetz). \( \gamma^p \) is bijective.

Question: Is Hard Lefshetz true for compact symplectic manifolds. If not, when is it true.

Define \( [L^i, L^j] = A \), by Kaehler-Weil says that \( A \alpha = (n - p) \alpha \).

Lemma. \( [A, L^i] = 2L^i \).

Proof. \( AL^i \alpha - L^i A\alpha = (n - (p - 2))L^i \alpha - (n - p)L^i \alpha = 2L^i \alpha \)

Lemma. \( [A, L] = -2L \).

There is another place in the world where you encounter these: Lie Groups.

Lie Groups

Take \( G = SL(2, \mathbb{R}) \), then consider the lie algebra \( \mathfrak{g} = sl(2, \mathbb{R}) \).

This is the algebra \( \{ A \in M_{22}(\mathbb{R}), tr A = 0 \} \). Generated by

\[
X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Check that \( [X, Y] = H, \ [H, X] = 2X \) and \( [H, Y] = -2Y \), and \( sl(2, \mathbb{R}) = span\{X, Y, Z\} \), and the above describes the Lie Algebra structure.

\( \rho : \mathfrak{g} \to End(\Omega) \) be given by \( X \mapsto L^i, Y \mapsto L \) and \( H \mapsto A \) is a representation of the Lie algebra \( \mathfrak{g} \) on \( \Omega \). So \( \Omega \) is a \( \mathfrak{g} \)-module.

Lemma. \( \Omega_{harm} \) is a \( \mathfrak{g} \)-module of \( \Omega \).

Proof. First note that \( Ld = dL, \) i.e. \( dL\alpha = d(\omega \wedge \alpha) = \omega \wedge d\alpha = Ld\alpha \). Taking transposes we get \( L^i \delta = \delta L^i \).

Then take \( \alpha \in \Omega_{harm} \). We already know that \( [d, L^i] = \delta \), so \( dL^i \alpha - L^i d\alpha = \delta \alpha \), which implies that \( dL^i \alpha = 0 \).

Similarly \( d\alpha, \delta \alpha = 0 \), so \( L\alpha, L^i \alpha \) are in \( \Omega_{harm} \).

So since \( A = [L, L^i], A\alpha \in \Omega_{harm} \) and \( \Omega \) is a \( \mathfrak{g} \)-module.
Note that $\Omega_{\text{harm}}$ is not finite dimensional. So these representations are not necessarily easy to deal with.

**Definition.** Let $V$ be a $\mathfrak{g}$-module. $V$ is of **finite $H$-type** if

$$V = \bigoplus_{i=1}^{N} V_i$$

and $H = \lambda_i \text{Id}$ on $V_i$.

In other words, $H$ is in diagonal form with respect to this decomposition.

**Example.** $\Omega = \bigoplus_{p=0}^{2n} \Omega^p$, $H = (n - p) \text{Id}$ on $\Omega^p$ and $\Omega_{\text{harm}} = \bigoplus_{p=0}^{2n} \Omega^p_{\text{harm}}$, $H = (n - p) \text{Id}$ on $\Omega^p_{\text{harm}}$.

**Theorem.** If $V$ is a $\mathfrak{g}$-module of finite type, then every sub and quotient module is of finite type.

**Proof.** $V = \bigoplus_{i=1}^{N} V_i$, $H = \lambda_i \text{Id}$ on $V_i$. Let $\pi_i : V \to V_i$ be a projection onto $V_i$. Check that

$$\pi_i = \frac{1}{\prod_{i \neq j} (\lambda_i - \lambda_j)} \prod_{j \neq i} (H - \lambda_j)$$

i.e., $\pi_i v = v$ on $v_i$. So $\pi_i$ takes sub/quotient objects onto themselves.