Lecture 26

Lemma. Take \( v \in V \), \( Hv = \lambda v \). We claim that \( H(Xv) = (\lambda + 2)Xv \).

Proof. \( (HX - XH)v = 2Xv \), so \( HXv = \lambda Xv + 2Xv = (\lambda + 2)Xv \).

Lemma. If \( Hv = \lambda v \), then
\[
[X,Y^k]v = k(\lambda - (k - 1))Y^{k-1}v
\]

Proof. We proceed by induction. If \( k = 1 \) this is just \( [X,Y]v = Hv = \lambda v \). This is true.

Now we show that if this is true for \( k \), its true for \( k + 1 \).

\[
[X,Y^{k+1}]v = XY^{k+1}v - Y^{k+1}Xv
\]

\[
= (XY)Y^k v - (YX)Y^k v + Y(XX^k)v - Y(Y^k Xv)
\]

\[
= HY^k v + Y([X,Y^k])v
\]

\[
= (\lambda - 2k)Y^k v + Y(k(\lambda - (k - 1)))Y^{k-1}v
\]

\[
= ((\lambda - 2k) + k(\lambda - k - 1))Y^k v = (k+1)(\lambda - k)Y^k v
\]

Definition. \( V \) is a cyclic module with generator \( v \) if every submodule of \( V \) containing \( v \) is equal to \( V \) itself.

Theorem. If \( V \) is a cyclic module of finite \( H \) type then \( \dim V < \infty \).

Proof. Let \( v \) generate \( V \). Then \( v = \sum_{i=0}^{N} v_i \) where \( v_i \in V_i \). It is enough to prove the theorem for cyclic modules generated by \( v_i \). We can assume without loss of generality that \( Hv = \lambda v \).

Now, note that only a finite number of expression \( Y^k X^i v \) are non-zero (since \( X \) shifts into a different eigenspace, and there are only a finite number of eigenspaces).

By the formula that we just proved, \( \text{span} \{ Y^k X^i v \} \) is a submodule of \( V \) containing \( v \).

Fact: Every finite dimensional \( g \)-module is a direct sum of irreducibles.

In particular, every cyclic submodule of \( V \) is a direct sum of irreducibles.

Theorem. Every irreducible \( g \)-module of finite \( H \) type is of the form \( V = V_0 \oplus \cdots \oplus V_k \) where \( \dim V_i = 1 \). Moreover, there exists \( v_i \in V_i \) \( \{0\} \) such that

\[
H v_i = (k - 2i) v_i
\]

\[
Y v_i = v_{i+1} \quad i \leq k - 1
\]

\[
X v_i = i(k - (i - 1)) v_{i-1} \quad i \geq 1
\]

\[
X v_0 = 0, Y v_k = 0
\]

Proof. Let \( V = V_0 \oplus \cdots \oplus V_n \), and \( H = \lambda I_d \) on \( V_i \) and assume that \( \lambda_0 > \lambda_1 > \cdots > \lambda_n \). Take \( v \in V_0 \) \( \{0\} \).

Note that \( Xv = 0 \), because \( HXv = (\lambda_0 + 2)Xv \) and \( \lambda_0 + 2 > \lambda_0 \).

Consider \( Yv, \ldots, Y^k v \neq 0 \), \( Y^{k+1} v = 0 \), so \( HY^i v = (\lambda_0 - 2i)Y^i v \). and

\[
XY^i v = Y^i Xv + i(\lambda - (i - 1))Y^{i-1}v = i(\lambda - (i - 1))Y^{i-1}v
\]

When \( i = k + 1 \) we have

\[
XY^{k+1} v = 0 = (k + 1)(\lambda - k)Y^k v
\]

but \( Y^k v \neq 0 \), so it must be that \( \lambda = k \). Now just set \( v_i = Y^i v \).
Lemma. Let $V$ be a $k + 1$ dimensional vector space with basis $v_0, \ldots, v_k$. Then the relations in the above theorem define an irreducible representation of $\mathfrak{g}$ on $V$.

Definition. $V$ a $\mathfrak{g}$-module, $V = \bigoplus_{i=0}^{N} V_i$ of finite H-type. Then $v \in V$ is primitive if

(a) $v$ is homogenous, (i.e. $v \in V_i$)
(b) $Xv = 0$.

Theorem. If $v$ is primitive then the cyclic submodule generated by $v$ is irreducible and $Hv = k$ where $k$ is the dimension of this module.

Proof. $v, Yv, \ldots, Y^k v \neq 0$, $Y^{k+1} = 0$. Take $v_i = Y^i v$. Check that $v_i$ satisfies the conditions.

Theorem. Every vector $v \in V$ can be written as a finite sum

$$v = \sum Y^l v_l$$

where $v_l$ is primitive.

Proof. This is clearly true if $V$ is irreducible (by the relations). Hence this is true for cyclic modules, because they are direct sums of irreducibles, hence this is true in general.

Corollary. The eigenvalues of $H$ are integers.

Proof. We need to check this for eigenvectors of the form $Y^l v$ where $v$ is primitive. But for $v$ primitive we know the theorem is true, i.e. $Hv = kv$, $HY^l v = (k - 2l)Y^l v$. So write $V = \bigoplus V_r$, $H = rId$ on $V_r$. 

$\square$