Chapter 6

Geometric Invariant Theory

Lecture 31

Lie Groups

Goof references for this material: Abraham-Marsden, Foundations of Mechanics (2nd edition) and Ana Canas p. 128

Let $G$ be a lie group. Denote by $\mathfrak{g}$ the Lie algebra of $G$ which is $T_e G$, with the lie bracket operation.

**Definition.** The exponential is a map $\exp : \mathfrak{g} \to G$ with the following properties

(a) $\mathbb{R} \to G, t \mapsto \exp t v$ is a lie group homomorphism.

(b) $d \left( \exp tv \right)_{t=0} = v \in T_e G = \mathfrak{g}$

**Example.** $G = GL(n, \mathbb{R}) = \{ A \in M_{n \times n}(\mathbb{R}) \mid \det(A) \neq 0 \}$. Then $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) = M_{n \times n}(\mathbb{R})$ and $[A, B] = AB - BA$ and

$\exp A = \sum \frac{A^i}{i!}$

**Example.** $G$ a compact connected abelian Lie group. Then the lie algebra is $\mathfrak{g}$ with $[,] = 0$. $\mathfrak{g}$ is a vector space, i.e. an abelian lie group in its own right. Then the exponential map $\exp : \mathfrak{g} \to G$ is a surjective lie group homomorphism.

Let $Z_G = \ker \exp$ be called the Group lattice of $G$, then $G = \mathfrak{g}/Z_G$, by the first isomorphism theorem.

For instance, take $G = (S^1)^n = T^n$, then $\mathfrak{g} = \mathbb{R}^n$, exp : $\mathbb{R}^n \to T^n$ is given by $(t_1, \ldots, t_n) \mapsto (e^{it_1}, \ldots, e^{it_n})$. Then $Z_G = 2\pi \mathbb{Z}^n$ and $G \cong \mathbb{R}^n/2\pi \mathbb{Z}^n$.

Group actions

Let $M$ be a manifold.

**Definition.** An action of $G$ on $M$ is a group homomorphism

$\tau : G \to Diff(M)$

where $\tau$ is smooth if $ev : G \times M \to M, (g, m) \mapsto \tau_g(m)$ is smooth.

**Definition.** Then infinitesimal action of $\mathfrak{g}$ on $M$

$d\tau : \mathfrak{g} \to Vect(M) \quad v \in \mathfrak{g} \mapsto v_M$

is given by

$\tau(\exp tv) = \exp(-tv_M)$
**Theorem.** $d\tau$ is a morphism of Lie algebras.

Given $p \in M$ denote

$$G_p = \{ g \in G, \tau_g(p) = p \}$$

This is the isotropy group of $p$ of the stabilizer of $p$. Then

$$\text{Lie} G_p = \{ v \in \mathfrak{g} \mid v_m(p) = 0 \}$$

**Definition.** The orbit of $G$ through $p$ is

$$G \circ p = \{ \tau_g(p) \mid g \in G \}$$

This is an immersed submanifold of $M$, and its tangent space is given by $T_p(G \circ p) = \mathfrak{g}/\mathfrak{g}_p$. The orbit space of $\tau$ is $M/G = \{ \text{set of all orbits, or equivalently } M/\sim \}$ where $p, q \in M$ and $p \sim q$ iff $p = \tau_g(q)$ for some $g \in G$.

We can topologize this space, by the projection

$$\pi : M \to M/G \quad p \mapsto G \circ p$$

and define the topology of $M/G$ by $U \subset M/G$ is open if and only if $\pi^{-1}(U)$ is open (i.e. assign $M/G$ the weakest topology that makes $\pi$ continuous). This, however, can be a nasty topological space.

**Example.** $M = \mathbb{R}$, $G = (\mathbb{R}^+, \times)$. And $\tau$ maps $t$ to multiplication by $t$. Then $M/G$ is composed of 3 points, $\pi(0), \pi(1)$ and $\pi(-1)$, but the set $\{\pi(1), \pi(-1)\}$ is not closed.

**Definition.** The action $\tau$ is free if $G_p = \{ e \}$ for all $p$ ($e$ the identity).

**Definition.** The action $\tau$ is locally free if $\mathfrak{g} = \{ 0 \}$ for all $p$ (this happens if and only if $G_p$ is discrete).

**Definition.** $\tau$ is a proper action if the map $G \times M \to M \times M$ given by $(g, m) \mapsto (m, \tau_g(m))$ is a proper map.

**Theorem.** If $\tau$ is free and proper then $M/G$ is a differentiable manifold and $\pi : M \to M/G$ is a smooth fibration.

**Proof.** (Sketch) S a slice of a $G$-orbit through $p$, i.e., $S$ is a submanifold of $M$ of codim = dim $G$, with $S \cap G \circ p = \{ p \}$, $T_p S \oplus T_p G \circ p = T_p M$. It's not hard to construct such slices.

Then look at the map $G \times S \to M$, $(g, s) \mapsto \tau_g(s)$. This is locally a diffeomorphism at $(e, p)$ and group invariance implies that it is locally a diffeomorphism on $G \times \{ p \}$. So it maps a neighborhood $W$ of $G \times \{ p \}$ diffeomorphically onto an open set $O \subseteq M$.

Properness insures that $W = G \times U_0$ where $(U_0, x_1, \ldots, x_n)$ is a coordinate patch on $S$ centered at $p$.

Let $U = O / G \cong U_0$ and $(U, x_1, \ldots, x_n)$ is a coordinate patch on $M/G$.

We claim that any two such coordinate patches are compatible (Maybe add a figure here?)

**Definition.** $G$ is a complex Lie group if $G$ is a complex manifold and the group operations $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are holomorphic.

**Example.** (a) $G = GL(n, \mathbb{C}) = \{ A \in M_n(\mathbb{C}) \mid \det A \neq 0 \}$. And the Lie algebra is $M_n(\mathbb{C}) = \mathfrak{gl}(n, \mathbb{C})$.

(b) $\mathbb{C}^* = \mathbb{C} - \{ 0 \}$.

(c) Complex Tori. For instance $T^n = (\mathbb{C}^*)^n$.

**Definition.** An action $\tau$ of $G$ on $M$ is holomorphic if

$$ev : G \times M \to M$$

is holomorphic.

In particular for $g \in G$, $\tau_g : M \to M$ is a biholomorphism and the $G$-orbits

$$G \circ p$$

are complex submanifolds of $G$.

**Theorem.** If $\tau$ is free and proper the orbit space $M/G$ is a complex manifold and the fibration $\pi : M \to M/G$ is a holomorphic fiber mapping.

**Proof.** Imitate the proof above with $S$ being a holomorphic slice of $G \circ p$ at $p$. }
Symplectic Manifolds and Hamiltonian $G$-actions

Let $G$ be a connected Lie group and $M, \omega$ a symplectic manifold. An action, $\tau$ of $G$ on $M$ is symplectic if $\tau^*_g \omega = \omega$ for all $g$, i.e. the $\tau_g$ are symplectomorphisms.

Thus if $v \in \mathfrak{g}$

$$\tau(\exp tv)^* \omega = \exp(-tv_M)^* \omega$$

Then

$$\frac{d}{dt} \exp(-tv_M)^* \omega \bigg|_{t=0} = L_{v_M} \omega = 0.$$ 

This implies that

$$\iota(v_M)d\omega + d\iota(v_M)\omega = d\iota(v_M)\omega = 0$$

so $\iota_{v_M}\omega$ is closed.

**Definition.** $\tau$ is a Hamiltonian action if for all $v \in \mathfrak{g}$, $\iota(v_M)\omega$ is exact.

**The Moment Map**

Choose a basis $v^1, \ldots, v^n$ of $\mathfrak{g}$ and let $v^*_1, \ldots, v^*_n$ be a dual basis of $\mathfrak{g}^*$.

If $\tau$ is hamiltonian then $\iota(v^*_M)\omega = d\phi^i$, where $\phi^i \in C^\infty(M)$.

**Definition.** The map $\Phi : M \to \mathfrak{g}^*$ defined by

$$\Phi = \sum \phi^i v^*_i$$

is called the moment map.

**Remarks**

(a) Note that for every $v \in \mathfrak{g}$,

$$\iota(v_M)\omega = d\phi^v \quad \text{where} \quad \phi^v = \langle \Phi, v \rangle$$

(b) $\Phi$ is only well defined up to an additive constant $c \in \mathfrak{g}^*$.

(c) If $M$ is compact one can normalize this constant by requiring that

$$\int_M \phi^i \omega^n = 0$$

(d) Another normalization: If $p \in M^G$, i.e. if $G_p = G$, then one can require that $\phi^i(p) = 0$ for $i = 1, \ldots, n$, then $\Phi(p) = 0$. 