Lecture 33

First, some general Lie theory things. $G$ a compact, connected Lie group. Let $G_C \supset G$ a complex Lie group.

**Definition.** $G_C$ is the complexification of $G$ if

(a) $g_C = \text{Lie } G_C = g \otimes \mathbb{C}$

(b) The complex structure on $T_e G_C$ is the standard complex structure on $g \otimes \mathbb{C}$.

(c) $\exp : g_C \rightarrow G_C$ maps $g$ into $G$.

(d) The map $\sqrt{-1} g \times G \rightarrow G_C$ defined by $(\omega, g) \mapsto (\exp \omega)g$ is a diffeomorphism.

Take $G = U(n)$. What is $g$? Let $H_n$ be the Hermitian matrices. If $A \in H_n$, then $\exp \sqrt{-1} t A \subset U(n)$, so $g = \sqrt{-1} H_n$.

**Exercise** Show $G_C = GL(n, \mathbb{C})$.

**Hints:**

(a) $M_n(\mathbb{C}) = \text{Lie } GL(n, \mathbb{C}) = H_n \oplus \sqrt{-1} H_n$ given by the decomposition

$$A \mapsto \frac{A + A^t}{2} + \frac{A - A^t}{2}$$

(b) Polar decomposition theorem: For $A \in GL(n, \mathbb{C})$ then $A = BC$ where $B$ is positive definite, $B \in H^n$ and $C \in U(n)$.

(c) $\exp : H_n^* \rightarrow H_n^\text{pos. def}$ is an isomorphism. This maps a matrix with eigenvalues $\lambda_i$ to a matrix with eigenvalues $e^{\lambda_i}$.

**Example.** Take $G$ a compact, connected abelian Lie group. Then $G = g / \mathbb{Z}_G$ and $G_C = g_C / \mathbb{Z}_G$.

Let $M$ be a Kaehler manifold, $\omega$ a Kaehler form, and $\tau$ a holomorphic action of $G_C$ on $M$.

**Definition.** $\tau$ is a **Kaehler action** if $\tau |_G$ is hamiltonian.

So we have a moment map $\Phi : M \rightarrow g^*$ and for $v \in g$ we have $v_M$ a vector field on $M$, and

$$\iota(v_M) \omega = d\phi^v \quad \phi^v = (\Phi, v)$$

For $p \in M$ note that because $M$ is Kaehler we have the addition bits of structure $(B_r)_p, (B_s)_p, J_p$ on $T_p M$.

Now take $v \in g$, $\sqrt{-1} v = w \in g_C$. From these we get corresponding vector fields $v_M, w_M$.

**Lemma.** At every $p \in M$

$$w_M(p) = J_p v_M(p)$$

**Proof.** Consider $\epsilon : G_C \rightarrow M, g \mapsto \tau_{g^{-1}}(p)$. This is a holomorphic map and $(de)_p : g_C \rightarrow T_p M$ is $\mathbb{C}$-linear and maps $v, w$ into $v_M(p), w_M(p)$.

**Proposition.** If $v \in g$, $w = \sqrt{-1} v$, then the vector field $w_M$ is the **Riemannian gradient** of $\phi^v$.

**Proof.** Take $p \in M, v \in T_p M$. Then

$$(B_r)_p(v, w_M(p)) = B_s(v, J_p w_M(p)) = -B_s(v, J_p v_M(p)) = \iota(v_M(p)) \omega_p(v) = d\phi^v_p(v)$$

QED
Assume $\Phi : M \to \mathfrak{g}^*$ is proper. Let $Z = \Phi^{-1}(0)$. Assume that $G$ acts freely on $Z$. Then $Z$ is a compact submanifold of $M$. Then we can form the reduction $M_{\text{red}} = Z/G$.

Consider $G_{\mathfrak{c}} \times Z \to M$ given by $(g, z) \mapsto \tau_g(z)$. Let $M_{\text{st}}$ be the image of this map. Note that $G_{\mathfrak{c}}$ is a subset of $M$.

**Theorem (Main Theorem).**  
(a) $M_{\text{st}}$ is an open $G_{\mathfrak{c}}$-invariant subset of $M$.
(b) $G_{\mathfrak{c}}$ acts freely and properly on $M_{\text{st}}$.
(c) Every $G_{\mathfrak{c}}$ orbit in $M_{\text{st}}$ intersects $Z$ in a unique $G$-orbit.
(d) Hence $M_{\text{st}}/G_{\mathfrak{c}} = Z/G = M_{\text{red}}$.
(e) $\omega_{\text{red}}$ is Kaehler.

**Proof.** (a) Since $M_{\text{st}}$ is $G_{\mathfrak{c}}$-invariant it suffices to show that $M_{\text{st}}$ contains an open neighborhood of $Z$. Note that since $G_{\mathfrak{c}} = (\exp \sqrt{-1}g)G$ implies that $M_{\text{st}}$ is the image of

$$
\psi : \sqrt{-1}g \times Z \to M \quad (\omega, p) \mapsto (\exp w_m)(p)
$$

Hence it suffices to show that $\psi$ is a local diffeomorphism at all points $(0, p)$. Hence it suffices to show that $(d\psi)_{0, p}$ is bijective.

But $(d\psi)_{0, p} : T_p Z \to T_{p^*} Z$. So it suffices to finally prove that

**Lemma.** $(d\psi)_{0, p}$ maps $\sqrt{-1} g$ bijectively onto $(T_p Z)^{\perp}$ in $T_{p^*} M$.

**Proof.** Let $w = \sqrt{-1}v$ in $\sqrt{-1} g$, $v \in T_p Z$. Then

$$
B_t(v, w_M(p)) = d\varphi_t^w(v) = 0
$$

so $w_M(p) \perp T_p Z$.

(b) $G_{\mathfrak{c}}$ acts freely on $M_{\text{st}}$.

**Lemma.** If $p \in Z$ and $w \in \sqrt{-1} g - \{0\}$. Then $(\exp w_M)(p) \in Z$.

**Proof.** Let $w = \sqrt{-1}v$, $v \in \mathfrak{g}$, then $(\exp tw_M)(p)$ is an integral curve of a gradient vector field of $\varphi^w$. Now $\varphi^w(p) = 0$ so $\varphi^w(\exp tw_M)(p) > 0$ for $t > 0$ (since gradient vector fields are increasing). So $\varphi^w(\exp w_M)(p) > 0$ and so $\exp w_M(p) \notin Z$.

To show that $G_{\mathfrak{c}}$ acts freely on $M_{\text{st}}$ it suffices to show that $G_{\mathfrak{c}}$ acts freely at $p \in Z$. Let $a \in G_{\mathfrak{c}}$, $a = (\exp -w)g$, where $w \in \sqrt{-1}g, g \in G$. Suppose $a \in (G_{\mathfrak{c}})p$ then $(\exp w_M)(\tau_g(p)) = p$. But $\tau_g(p) = q \in Z$. So $(\exp w_M)(q) = p \in Z$ which implies $w = 0, a = G$. So $(G_{\mathfrak{c}}) = G_{\mathfrak{c}} = \{e\}$.

We will skip proving that $G_{\mathfrak{c}}$ acts properly on $M_{\text{st}}$.

(c) This will be an exercise

**Exercise** Every $G_{\mathfrak{c}}$-orbit in $M_{\text{st}}$ intersects $Z$ in a unique $G$ orbit. Hint: Every $G_{\mathfrak{c}}$ orbit in $M_{\text{st}}$ is of the form $G_{\mathfrak{c}} \circ p$ with $p \in Z$. $a \in (G_{\mathfrak{c}} \circ p) \cap Z$. Then $a = (\exp w_M)\tau_g(p), g \in G$, $w \in -\sqrt{-1}g$. Argue as before and force $w = 0$.

(d) So $M_{\text{red}} = Z/G = M_{\text{st}}/G_{\mathfrak{c}}$.

(e) All that remains to show is that $\omega_{\text{red}}$ is Kaehler.

**Proof.** $p \in Z, \pi : Z \to M_{\text{red}}, q = \pi(p)$. Let $V$ be the $B_r$-orthocomplement in $T_pM$ to $T_p(G_{\mathfrak{c}} \circ p)$ implies that $V \subseteq T_pZ$ and its perpendicular to $T_pG \circ p$.

Remember we have $d\pi : M_{\text{st}} \to M_{\text{red}} = M_{\text{st}}/G_{\mathfrak{c}}$ is a holomorphic action.

So $d\pi_p : V \to T_q M_{\text{red}}$ is $\mathbb{C}$-linear and $\omega_p | V = (d\pi_p)^* \omega_{\text{red}} | V$, where $V$ a complementary subspace of $T_p M$ so $\omega_p |$ is Kaehler implies that $(\omega_{\text{red}})_q$ is Kaehler.

\[\Box\]