Lecture 34

Let $G$ be an $n$-dimensional compact connected abelian Lie group. Let $\mathfrak{g}$ be the Lie algebra of $G$.

For an abelian Lie group $\exp : \mathfrak{g} \to G$ is a group epi-morphism and $\mathbb{Z}_G = \ker \exp$ is called the group lattice of $G$. Since $\exp$ is an epi-morphisms, $G = \mathfrak{g}/\mathbb{Z}_G$. So we can think of $\exp : \mathfrak{g} \to G$ as a projection $\mathfrak{g} \to \mathfrak{g}/\mathbb{Z}_G$.

Representations of $G$

We introduce the dual lattice $\mathbb{Z}_G^* \subseteq \mathfrak{g}^*$ a weight lattice, with $\alpha \in \mathfrak{g}^*$ in $\mathbb{Z}_G^*$ if and only if $\alpha(v) \in 2\pi \mathbb{Z}$ for all $v \in \mathbb{Z}_G$.

Suppose we’re given $\alpha_i \in \mathbb{Z} st_G$, $i = 1, \ldots, d$. We can define a homomorphism $\tau : G \to GL(d, \mathbb{C})$ by

$$\tau(\exp v)z = (e^{\sqrt{-1}\alpha_1(v)}z_1, \ldots, e^{\sqrt{-1}\alpha_d(v)}z_d)$$

and this is well-defined, because if $v \in \mathbb{Z}_G$, $\tau(\exp v) = 1$. But think of $\tau$ as an action of $G$ on $\mathbb{C}^d$. We get a corresponding infinitesimal actions

$$d\tau : \mathfrak{g} \to \mathcal{X}(G) \quad v \mapsto v_{\mathbb{C}^d} \quad d\tau(-tv) = \exp tv_{\mathbb{C}^d}.$$

We want a formula for this. We introduce the coordinates $z_i = x_i + \sqrt{-1}y_i$. We claim

$$(\Pi) \quad v_{\mathbb{C}^d} = -\sum \alpha_i(v)\left(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i}\right).$$

We must check that for each coordinate $z_i$

$$\frac{d}{dt}(\tau_{\exp -tv})^*z_i \bigg|_{t=0} = L_{v_{\mathbb{C}^d}}z_i.$$

The LHS is

$$\frac{d}{dt}e^{-\sqrt{-1}\alpha_i(v)}z_i = -\alpha_i(v)z_i$$

and the RHS is

$$\left(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i}\right)(x_i + \sqrt{-1}y_i) = \sqrt{-1}z_i$$

so

$$L_{v_{\mathbb{C}^d}}z_i = \sqrt{-1}\alpha_i(v)z_i$$

Take $\omega$ to be the standard kaehler form on $\mathbb{C}^d$

$$\omega = \sqrt{-1} \sum dz_i \wedge d\bar{z}_i = 2 \sum dx_i \wedge dy_j$$

**Theorem.** $\tau$ is a Hamiltonian action with moment map

$$\Phi : \mathbb{C}^d \to \mathfrak{g}^*$$

where

$$\Phi(z) = \sum |z_i|^2 dz_i$$
Proof.

\[ \iota(v_{\mathbb{C}^d})\omega = \left( - \sum \alpha_i(v) \left( x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} \right) \right) \wedge \sum dx_i \wedge dy_i \]
\[ = 2 \sum \alpha_i(v) x_i dx_i + y_i dy_i = \sum \alpha_i(v) d(x_i^2 + y_i^2) \]
\[ = d \sum \alpha_i(v) |z_i|^2 = d(\Phi, v) \]

N.B. \( \Phi(0) = 0, 0 \in (\mathbb{C}^d)^G \) implies that \( \Phi \) is an equivariant moment map. \( \square \)

**Definition.** \( \alpha_1, \ldots, \alpha_d \) are said to be polarized if for all \( v \in \mathfrak{g} \) we have \( \alpha_i(v) > 0 \).

**Theorem.** If \( \alpha_1, \ldots, \alpha_d \) are polarized then \( \Phi : \mathbb{C}^d \to \mathfrak{g}^* \) is proper.

**Proof.** The map \( (\Phi, v) : \mathbb{C}^d \to \mathbb{R} \) is already proper if \( \alpha_i(v) > 0 \), so the moment map itself is proper. \( \square \)

Now, given \( z \in \mathbb{C}^d \), what can be said about \( G_z \) and \( \mathfrak{g}_z \)?

**Notation.** \( I_z = \{i, z_i \neq 0\} \)

**Theorem.**

(a) \( G_z = \{ \exp v \mid \alpha_i(v) \in 2\pi\mathbb{Z} \text{ for all } i \in I_z\} \)

(b) \( \mathfrak{g}_z = \{v \mid \alpha_i(v) = 0 \text{ for all } i \in I\} \)

**Corollary.** \( \tau \) is locally free at \( z \) if and only if \( \operatorname{span}_\mathbb{R}\{\alpha_i, i \in I_z\} = \mathfrak{g}^* \). \( \tau \) is free at \( z \) if and only if \( \operatorname{span}_\mathbb{R}\{\alpha_i, i \in I_z\} = \mathbb{Z}^*_G \).

Let \( a \in \mathfrak{g}^* \). Is \( a \) a regular value of \( \Phi \)?

**Notation.**

\[ \mathbb{R}^d_+ = \{ (t_1, \ldots, t_d) \in \mathbb{R}^d, t_i \geq 0 \} \]
\[ I \subset \{1, \ldots, d\} \quad (\mathbb{R}^d_+)_I = \{ t \in \mathbb{R}^d_+, t_i > 0 \iff i \in I \} \]

Consider \( L : \mathbb{R}^d_+ \to \mathfrak{g}^* \)

\[ L(t) = \sum t_i \alpha_i \]

Assume \( \alpha_i \)'s are polarized. \( L \) is proper. Take \( a \in \mathfrak{g}^* \). Let \( \Delta_a = L^{-1}(a) \), then \( \Delta_a \) is a convex polytope. Denote \( \mathcal{I}_\Delta_a = \{I, (\mathbb{R}^d_+)_I \cap \Delta_a \neq \emptyset \} \). For \( I \in \mathcal{I}_\Delta \) we have that \( (\mathbb{R}^d_+)_I \cap \Delta = \text{the faces of } \Delta \).

**Theorem.** \( a \in \mathfrak{g}^* \) is a regular value of \( \Phi \) if and only if for all \( I \in \mathcal{I}_\Delta \) we have \( \operatorname{span}_\mathbb{R}\{\alpha_i, i \in I\} = \mathfrak{g}^* \) and \( G \) acts freely on \( \Phi^{-1}(a) \) if and only if \( \operatorname{span}_\mathbb{Z}\{\alpha_i, i \in I\} = \mathbb{Z}^*_G \).

**Proof.** \( \Phi \) is the composite of \( L : \mathbb{R}^d_+ \to \mathfrak{g}^* \) and the map \( \gamma : \mathbb{C}^d \to \mathbb{R}^d_+ \) which maps \( z \mapsto (|z_1|^2, \ldots, |z_d|^2) \) so \( z \in \Phi^{-1}(a) \) if an only if \( \gamma(z) \in \Delta_a \). How just apply above. \( \square \)

**Symplectic Reduction**

Take \( a \in \mathfrak{g}^* \). Suppose \( a \) is a regular value of \( \Phi \), i.e. \( \mathfrak{g}_z = \{0\} \) for all \( z \in \Phi^{-1}(a) \). Then \( Z_a = \Phi^{-1}(a) \) is a compact submanifold of \( \mathbb{C}^d \).

Suppose \( G \) acts freely on \( Z_a \). Then \( M_a = Z_a / G \). Consider \( i : Z_a \to \mathbb{C}, \pi : Z_a \to M_a \).

**Theorem.** There exists a unique symplectic form \( \omega_a \) on \( M_a \) such that \( \pi^* \omega_a = i^* \omega_a \).

**Proof.** Apply the symplectic quotient procedure to \( \Phi^{-1}(a) \). \( \square \)

Let \( G_C = \mathfrak{g}_C / Z_G = \mathfrak{g} \otimes \mathbb{C} / \mathfrak{g}_Z \). By \( (I) \), \( \tau \) extends to a holomorphic action of \( G_C \) on \( \mathbb{C}^d \). Then \( G_C \cdot \Phi^{-1}(a) = \{ \tau_g(z) \mid g \in G_C, z \in Z_a \} = \mathbb{C}^d_{\text{stable}}(a) \)
then \( M_a = \mathbb{C}^d_{\text{stable}}(a) / G_C = \text{the holomorphic description of } M_a. \omega_a \text{ is Kaehler. This } M_a \text{ is a toric variety.} \)

**Theorem.**

\[ \mathbb{C}^d_{\text{stable}}(a) = \bigcup_{I \in \mathcal{I}_\Delta} \mathbb{C}^d_I \]
where \( \mathbb{C}^d_I = \{ z \in \mathbb{C}^d \mid I_z = I \} \)