Lecture 36

Let \( G \) be an \( n \)-torus, and \( \alpha_1, \ldots, \alpha_d \in \mathbb{Z}^*_G \). Define a Hamiltonian action \( \tau \) of \( G \) on \( \mathbb{C}^d \) as follows. First we have

\[
L : \mathbb{R}^d \to \mathfrak{g}^* \quad L(t) = \sum t_i \alpha_i
\]

and

\[
\gamma : \mathbb{C}^d \to \mathbb{R}^d \quad \gamma(z) = (|z_1|^2, \ldots, |z_d|^2)
\]

then \( \Phi = L \circ \gamma \) is the moment map of \( \tau \). As before, we’re interested in the regular values of \( \Phi \).

Define \( \Delta_a = L^{-1}(a) \cap \mathbb{R}^d_+ \) a convex polytope.

**Theorem (1).** \( a \) is a regular value if \( \Delta_a \) is a simple \( n \)-dimensional.

For a regular call \( Z_a = \Phi^{-1}(a) \). Assume \( G \) acts freely on \( Z_a \). we have \( M_a = Z_a/G \).

\[
\begin{array}{c}
\xymatrix{ Z_a \ar[r]^i & \mathbb{C}^d \\
\psi \ar[u] & M_a \ar[l]_\pi}
\end{array}
\]

\( \psi : M_a \to \mathbb{R}^d \) and \( \psi \circ \pi = \gamma \circ i \).

\( Z_a = \gamma^{-1}(\Delta_a) \) implies that \( \psi(M_a) = \Delta_a \).

**Definition.** \( \Delta_a \) is called the **moment polytope**

For \( \xi \in \mathbb{R}^d \), let \( f = \langle \psi, \xi \rangle \) and \( \pi^* f = i^* f_0 \) where

\[
f_0(z) = \sum_{i=1}^d \xi_j |z_j|^2
\]

**Theorem (2).** Suppose that for all adjacent \( v, v' \) of \( \Delta_a \) we have \( \langle v - v', \xi \rangle \neq 0 \). Then

(a) \( f \) is Morse

(b) \( \psi \) maps \( \text{Crit}(f) \) bijectively onto \( \text{Vert}(\Delta_a) \).

(c) For \( q \in \text{Crit}(f) \) \( \text{ind}_q = \text{ind}_v \) where \( v = \psi(a) \) and the index \( \text{ind}_v \xi \) is given by

\[
\text{ind}_v \xi = \{ v_k \mid \langle v_k - v, \xi \rangle < 0 \}
\]

where the \( v_k \)'s are vertices adjacent to \( v \).

Recall:

\( I \subseteq \{1, \ldots, d\} \) then \( t \in \mathbb{R}^d_+ \) if and only if \( t_i \neq 0 \) if and only if \( i \in I \). For \( \Delta = \Delta_a \)

\[
\mathcal{I}_\Delta = \{ I \subseteq \mathbb{R}^d_+ \cap \Delta \neq \emptyset \}
\]

For \( I \in \mathcal{I}_\Delta \), \( \Delta_I = \mathbb{R}^d_+ \cap \Delta = \text{faces of the polytope } \Delta \). Recall also that there is a partial ordering \( I_1 \leq I_2 \) if and only if \( I_1 \subseteq I_2 \).

For \( I \) minimal \( \Delta_I = \{ v_I \} \)

**Theorem.** \( a \) is a regular value if and only if for every vertex \( v_I \) of \( \Delta_a \), \( a_i, i \in I \) form a basis of \( \mathfrak{g}^* \).

Let \( v_I \in \text{Vert}(\Delta_a) \). Relabel \( I = (1, 2, \ldots, n) \) so that \( \alpha_1, \ldots, \alpha_n \) are a basis for \( \mathfrak{g}^* \), \( a = \sum_{i=1}^n a_i \alpha_i \), \( L(v_I) = a \). \( v_I = (a_1, \ldots, a_n, 0, \ldots, 0) \) and for \( k > n \),

\[
\alpha_k = \sum a_{k,i} \alpha_i
\]

Rewrite

\[
L(t) = \sum_{i=1}^n \left( t_i - \sum_{k>n} a_{k,i} t_k \right) \alpha_i = \sum a_i \alpha_i = a
\]

From this we conclude that \( \Delta_a \) is defined by

\[
(I) \quad \begin{cases} 
t_i = a_i = \sum a_{k,i} t_k \\
t_1, \ldots, t_d \geq 0
\end{cases}
\]

We see immediately tat \( \Delta_a \) is \( m \)-dimensional, \( m = d - n \). The edges of \( \Delta_a \) at \( v_I \) lie along the rays \( v_I + se_k \), \( k = n + 1, \ldots, d \) for \( s \geq 0 \).

**Exercise** Check that \( e_k = (-a_{k,1}, \ldots, a_{k,n}, 0, \ldots, 1, \ldots, 0) \) where the 1 is in the \( k \)th slot.

The conclusion is that \( \Delta_a \) is simple at \( v_I \) so \( \Delta_a \) is simple.

Let \( v = v_I \) be a vertex of \( \Delta_a \). Write

\[
O_v = \{ t \in \Delta_a, t_i > 0 \text{ if } i \in I \} = \bigcup_{J \geq I} J \Delta_I.
\]
Consider $\gamma^{-1}(O_v)$. These are open $G$-invariant sets in $Z_a$
Take $U_v = \pi(\gamma^{-1}(O_v))$ an open cover of $M_a$. Let $f : M_a \to \mathbb{R}$. What does $f$ look like on $f|_{U_v}$. Take $I = (1, \ldots, n)$ by relabeling. Then

$$a = \sum_{i=1}^{n} a_i \alpha_i \quad v_I = (a_1, \ldots, a_n, 0, \ldots, 0)$$

then

$$z \in \gamma^{-1}(v_I) \iff \begin{cases} |z_i|^2 = a_i & i = 1, \ldots, n \\ z_k = 0 & k > n \end{cases}$$

**Proposition.** $\gamma^{-1}(v_I)$ is a single $G$-orbit.

**Proof.** $\dim \gamma^{-1}(v) = n$, $\dim G = n$ and $G$ acts freely on $\gamma^{-1}(v)$.

Moreover, $z \in Z_a$ if and only if $\gamma(z) \in \Delta_a$. Hence by (I) $O_v$ is defined by

$$|z_i|^2 = a_i - \sum a_{k,i} |z_k|^2$$

and $z_i \neq 0$, $i = 1, \ldots, n$.

Take $f_0 = \sum \xi_j |z_j|^2$ then (*)

$$i^* f_0 = c + \sum_{k>n} \left( \xi_k - \sum a_{k,i} \xi_i \right) |z_k|^2$$

$$= c + \sum_{k>n} \langle e_k, \xi \rangle |z_k|^2 = \pi^* f$$

where $e_k$ is defined as before. \hfill $\square$

**Proof of Theorem 2.** From (*) the only critical point of $f$ on $U_v$ is $a = \pi(\gamma^{-1}(v))$. (Recall $\gamma^{-1}(v)$ is a single $G$-orbit).

Moreover $\psi(a) = v_I$. Finally if $p \in \gamma^{-1}(v)$, then

$$(d\pi)_p^* (d^2 f_a) = \sum_{k>n} \langle e_k, \xi \rangle |z_k|^2 = \sum_{k>n} \langle e_k, \xi \rangle (x_k^2 + y_k^2)$$

It follows that $(d^2 f_a)$ is $(\ldots)$, and the index is $2 \text{ind} \xi v$. \hfill $\square$

Also a consequence

$$H^{2k+1}(M_a) = 0$$

so

$$b_k = \dim H^{2k}(M_a) = \# \{ \text{Vert} (\Delta_a), \text{ind} \xi v = k \}$$

and $b_k = \# \{ \text{ind} \xi iv = v \}$ doesn’t depend on $\xi$. If $f_k$ is the number of $k$-dimensional faces of $\Delta_a$ for $k = 0, \ldots, m$ then

$$f_{m-k} = \binom{m}{k} b_0 + \binom{m-1}{k-1} b_1 + \cdots + b_k$$

**Exercise** Prove this.

Let $\Delta$ be a simple $m$-dimensional convex polytope and $f_k$ be the number of $k$-dimensional faces of $\Delta$. Define $b_0, \ldots, b_n$ by the solutions to the equations

$$f_{m-k} = \binom{m}{k} b_0 + \ldots b_k$$
Then

**Theorem (McMullen, Stanley).** (a) The $b_k$s are integers.

(b) $b_{m-k} = b_k$

(c) $b_0 \leq b_1 \leq \cdots \leq b_k$ where $k = \left\lfloor \frac{m}{2} \right\rfloor$.

**Proof.** Exhibit $\Delta$ as the moment polytope of a toric variety of $M$.

(a) The $b_k$s are Betti numbers of $M$ (so integers)

(b) Poincare duality

(c) Hard Lefschetz.