Integration as a linear functional. A complex vector space is a set $V$ with two operations: addition (+) and scalar multiplication ($\cdot$).

Addition: For all $x, y, z \in V$,
- $x + y = y + x$.
- $x + (y + z) = (x + y) + z$.
- There exists a unique vector 0 such that $x + 0 = x$ for all $x$.
- There exists $(-x)$ such that $x + (-x) = 0$.

Multiplication: For all $\alpha, \beta \in \mathbb{C}$, $x \in V$,
- $x = x \cdot 1$.
- $x \cdot (\beta \cdot x) = (\alpha \beta) \cdot x$.
- $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$.
- $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$.

A linear transformation is a map $\Lambda : V_1 \to V_2$ from a vector space $V_1$ to a vector space $V_2$ such that $\Lambda(\alpha x + \beta y) = \alpha \Lambda x + \beta \Lambda y$. If $V_2 = \mathbb{C}$ (or $\mathbb{R}$), then $\Lambda$ is a linear functional.

Let $(X, \mathcal{M}, \mu)$ be a measure space. Then
\[
L^1(\mu) = \left\{ f : X \to \mathbb{C} \mid \int_X |f| \, d\mu < \infty, f \text{ measurable} \right\}.
\]

Note that $\int_X : f \mapsto \int_X f \, d\mu$ is a linear functional. Let $g : X \to \mathbb{C}$ be a bounded measurable function. Then $f \mapsto \int_X fg \, d\mu$ is also a linear functional.

Special case: $X = \mathbb{R}^n$. Let
\[
C(\mathbb{R}^n, \mathbb{R}) = \{ f : \mathbb{R}^n \to \mathbb{R} \mid f \text{ continuous} \}.
\]
The Riemann integral is a positive linear functional since $f \geq 0 \Rightarrow \Lambda f \geq 0$, where $\Lambda$ is the Riemann integral.

Riesz theorem. Let $X$ be a topological space and $C(X)$ be the set of functions from $X$ to $\mathbb{R}$. If $\Lambda : C \to \mathbb{R}$ is a positive linear functional, then there exists a $\sigma$-algebra $\mathcal{M}$ and unique measure $\mu$ on $X$ such that $\Lambda f = \int_X f \, d\mu$. Conversely, given a measure, then $\Lambda$ is a positive linear functional.

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Topology. Let $X$ be a topological space. The space $X$ is Hausdorff if for all $p, q \in X$ such that $p \neq q$ there exist neighborhoods $U$ and $V$ such that $p \in U$, $q \in V$, and $U \cap V = \emptyset$. The space $X$ is locally compact if for all $p \in X$ there exists a neighborhood $U$ of $p$ such that $\overline{U}$ (the closure of $U$) is compact. (Infinite dimensional spaces are not locally compact.)

Let $f : X \to \mathbb{R}$. If $\{x \mid f(x) > \alpha\}$ is open for all $\alpha$, then $f$ is lower semicontinuous. If $\{x \mid f(x) < \alpha\}$ is open for all $\alpha$, then $f$ is upper semicontinuous. Examples: $\chi_U$ for $U$ open is lower semicontinuous and $\chi_F$ for $F$ closed is upper semicontinuous.

The support of a function $f$ is defined as the set $\text{supp } f = \{x \mid f(x) \neq 0\}$. An important set is the set of all functions with compact support:

$$C_c(X) = \{f : X \to \mathbb{C} \mid \text{supp } f \text{ is compact}\}.$$ Since $\text{supp } f_g \subset (\text{supp } f) \cup (\text{supp } g)$, $C_c(X)$ is a vector space.

Notation: (1) $K \prec f$ means that $K$ is compact, $f \in C_c(X)$, $0 \leq f(x) \leq 1$ for all $x \in X$, and $f(x) = 1$ for all $x \in K$. (2) $f \prec V$ means that $V$ is open, $f \in C_c(X)$, $0 \leq f(x) \leq 1$ for all $x \in X$, and $\text{supp } f \subset V$.

Urysohn’s lemma. Let $X$ be a locally compact Hausdorff space, $K \subset V$, $K$ compact, $U$ open. Then there exists $f \in C_c(X)$ such that $K \prec f \prec V$.

A corollary to Urysohn’s lemma is the existence of partitions of unity. Let $V_1, \ldots, V_n$ be open subsets of $X$ (a locally compact Hausdorff space) and $K$ compact such that $K \subset V_1 \cup \cdots \cup V_n$. Then there exists functions $h_i \prec V_i$ such that $h_1(x) + \cdots + h_n(x) = 1$.

Riesz representation theorem (for positive linear functionals).

**Theorem 0.1.** Let $X$ be a locally compact Hausdorff space. Let

$$\Lambda : C_c(X) \to \mathbb{C}$$

be a positive linear functional (positive when restricted to $f : X \to \mathbb{R}_{\geq 0}$). Then there exists a $\sigma$-algebra $\mathcal{M}$ in $X$ which contains all the Borel sets and a unique positive measure $\mu$ on $\mathcal{M}$ such that

(a) $\Lambda f = \int_X f \ d\mu$ for all $f \in C_c(X)$.

(b) $\mu(K) < \infty$ for all compact sets $K \subset X$.

(c) If $E \in \mathcal{M}$, then

$$\mu(E) = \inf \{\mu(V) \mid E \subset V, V \text{ open}\}.$$ 

(d) If $E$ is open or $E \in \mathcal{M}$ with $\mu(E) < \infty$, then

$$\mu(E) = \sup \{\mu(K) \mid K \subset E, K \text{ compact}\}.$$
(e) If $E \in M$, $A \subset E$, and $\mu(E) = 0$, then $A \in M$.

**Proof.** (Outline) We must show uniqueness.

By (d), the measure of open sets determined by measure of compact sets, and so by (c) the measure of any set in $M$ is determined by the measure of compact sets. Assume we have $\mu_1$ and $\mu_2$ which satisfy the conditions of the theorem, and let $K$ be compact. For any $\epsilon > 0$, choose $U$ open such that $K \subset U$ and $\mu_2(U) < \mu_2(K) + \epsilon$. By Urysohn’s lemma, there exists $f \in C_c(X)$ such that $K \prec f \prec V$. Then

$$\mu_1(K) = \int_X \chi_K \, d\mu_1 \leq \int_X f \, d\mu_1 = \Lambda f$$

and

$$\Lambda f = \int_X f \, d\mu_2 \leq \int_X \chi_V \, d\mu_2 = \mu_2(V) < \mu_2(K) + \epsilon.$$

Since this holds for any $\epsilon > 0$, $\mu_1(K) \leq \mu_2(K)$, and by reversing the roles of $\mu_1$ and $\mu_2$, we have $\mu_1(K) = \mu_2(K)$.

Now let $V \subset X$ be open and define $\mu(V) = \sup\{\Lambda f \mid f \prec V\}$. For $E \subset X$, define $\mu(E) = \inf\{\mu(V) \mid E \subset V, V \text{ open}\} = \lambda^*(E)$. ($\lambda^*$ will not be countably additive on all sets, only on the $\sigma$-algebra.) Let $M_F$ be the set of $E \subset X$ such that

$$\mu(E) = \sup\{\mu(K) \mid K \subset E, K \text{ compact}\} \text{ and } \mu^*(E) < \infty.$$ 

Finally, $M$ is simply $E \subset X$ such that $E \cap K \in M_F$ for all $K \in M_F$. \hfill \Box

**Properties.**

1. $\mu^*$ is countably subadditive: $\mu(\bigcup E_i) \leq \sum \mu(E_i)$.
2. If $E_i \in M_F$ are disjoint, then $\mu(\bigcup E_i) = \sum \mu(E_i)$.
3. $M_F$ contains all open sets.
4. (Approximation) If $E \in M_F$ and $\epsilon > 0$, then there exist $K \subset E \subset V$, $K$ compact, $V$ open, such that $\mu(V \setminus K) < \epsilon$.
5. $M$ is a $\sigma$-algebra that contains the Borel $\sigma$-algebra $B$, and $\mu$ is countably additive on $M$.
6. If $f \in C_c(X)$, then $\Lambda f = \int_X f \, d\mu$.

**Proof.** Just NTS that $\Lambda f \leq \int_X f \, d\mu$ for $f$ real in $C_c(X)$. Then

$$-\Lambda f = \Lambda(-f) \leq \int_X (-f) \, d\mu = -\int_X f \, d\mu$$

$$\Rightarrow \Lambda f \geq \int_X f \, d\mu.$$

The complex case follows from the real case by complex linearity. Let $f \in C_c(X)$ and $\text{supp} \, f = K$ compact. The continuous
image of compact sets is compact \( \Rightarrow f(K) \subset [a, b] \). Choose 
\( \epsilon > 0 \) and choose \( y_i \) (\( i = 0, 1, \ldots, n \)) such that \( y_i - y_{i-1} < \epsilon \) and 
\( y_0 < a < y_1 < \cdots < y_n = b \) (i.e., partition the range by \( \epsilon \)). Let 
\[
E_i = \{ x \mid y_{i-1} < f(x) \leq y_i \} \cap K.
\]
Since \( f \) is continuous, \( f \) is Borel measurable and \( \bigcup_{i=1}^{n} E_i = K \) 
is a disjoint union. choose open sets \( V_i \supset E_i \) such that \( \mu(V_i) < 
\mu(E_i) + \epsilon/n \) for each \( i = 1, \ldots, n \) and \( f(x) < y_i + \epsilon \) for all \( x \in V_i \).
(The latter can be done by continuity of \( f \).)

By partition of unity, there exists \( h_i \prec V_i \) such that \( \sum_i h_i = 1 \)
on \( K \). Write \( f = \sum_i h_i f \). Then 
\[
\mu(K) \leq \Lambda(\sum_i h_i) = \sum_i \Lambda h_i,
\]
\[
h_i f \leq (y_i + \epsilon)h_i, \quad \text{and} \quad y_i - \epsilon < f(x) \quad \forall x \in E_i.
\]
Thus, 
\[
\Lambda f = \sum_{i=1}^{n} \Lambda(h_i f) \leq \sum_{i=1}^{n} (y_i + \epsilon)\Lambda h_i
\]
\[
= \sum_{i=1}^{n} (|a| + y_i + \epsilon)\Lambda h_i - |a| \sum_{i=1}^{n} \Lambda h_i
\]
\[
\leq \sum_{i=1}^{n} (|a| + y_i + \epsilon)(\mu(E_i) + \epsilon/n) - |a| \mu(K)
\]
\[
= \sum_{i=1}^{n} (|a| + \epsilon)(\mu(E_i))
\]
\[
= |a| \mu(K) + \sum_{i=1}^{n} (|a| + y_i + \epsilon)\epsilon/n + \sum_{i=1}^{n} y_i \mu(E_i)
\]
\[
= \sum_{i=1}^{n} (y_i - \epsilon)\mu(E_i) + 2\epsilon \mu(K) + \epsilon/n \sum_{i=1}^{n} (|a| + y_i + \epsilon)
\]
\[
\leq \int_X f \, d\mu + \epsilon(\text{constant}).
\]

Definitions. A measure space \((X, \mathcal{M}, \mu)\) is called a **Borel measure** if 
\( \mathcal{B} \subset \mathcal{M} \). If \( \mu(E) = \inf \{ \mu(V) \mid E \subset V, V \text{ open} \} \) for all \( E \in \mathcal{M}, \) 
then \( \mu \) is called **outer regular**. Similarly, if \( \mu(E) = \sup \{ \mu(K) \mid K \subset 
E, K \text{ compact} \} \) for all \( E \in \mathcal{M}, \) then \( \mu \) is called **inner regular**. If \( \mu \) is 
both inner and outer regular, it is said to be **regular**.
A space $X$ is $\sigma$-compact if $X = \bigcup_{i=1}^{\infty} K_i$ where each $K_i$ is compact. It is $\sigma$-finite if $X = \bigcup_{i=1}^{\infty} E_i$ where $\mu(E_i) < \infty$ for each $i$.

**Addition to Riesz.** If $X$ is locally compact, $\sigma$-compact, Hausdorff space then we also have:

1. If $E \in \mathcal{M}$ and $\epsilon > 0$, then there exists $F \subset E \subset V$, $F$ closed, $V$ open, such that $\mu(V \setminus F) < \epsilon$.
2. For all $E \in \mathcal{M}$ there exists $A \subset E \subset B$ such that $A$ is $F_\sigma$, $B$ is $G_\delta$, and $\mu(B \setminus A) = 0$.

**Application.** Let $X = \mathbb{R}^k$, $\Lambda : C_c(X) \to \mathbb{R}$ given by $\Lambda f = \int_X f$, the Riemann integral. Then Lebesgue measure is what you get from the Riesz theorem.