Appoximation of measurable functions by continuous functions. Recall Lusin’s theorem. Let $f: X \to \mathbb{C}$ be measurable, $A \subset X$, $\mu(A) < \infty$, and $f(x) = 0$ if $x \notin A$. Given $\epsilon > 0$, there exists $g \in C_c(X)$ such that $\mu(\{x \mid f(x) \neq g(x)\}) < \epsilon$ and

$$\sup_{x \in X} |g(x)| \leq \sup_{x \in X} |f(x)|.$$  

A corollary with the same assumptions and $f$ bounded (i.e., $|f(x)| < M$) is that there exists sequence $g_n \in C_c(X)$, $|g_n| < M$ such that $\lim g_n(x) = f(x)$ almost everywhere.

**Convergence almost everywhere.** Lebesgue’s dominated convergence theorem (LDCT) in the case of almost everywhere.

**Theorem 0.1.** Let $f_1, f_2, \ldots : X \to \mathbb{C}$ be a sequence of measurable functions defined a.e. Let $g: X \to \mathbb{C}$ be defined almost everywhere and $g \in L^1(\mu)$. Assume $\lim_{k \to \infty} f_k(x)$ exists for a.e. $x \in X$ and $|f_k(x)| \leq |g(x)|$ for a.e. $x \in X$. Then

$$\int_X \left( \lim_{k \to \infty} f_k \right) d\mu = \lim_{k \to \infty} \int_X f_k \, d\mu.$$  

**Proof.** Let $E_k = \{x \mid |f_k(x)| \geq |g(x)|\}$. Then $\mu(E_k) = 0$. Let $E = \bigcup_{k=1}^{\infty} E_k$. Then $\mu(E) = 0$. Redefine $f_k = 0$ on $E$; this does not change the integrals. Now $|f_k| \leq |g|$ a.e., and we can apply the regular LDCT. 

**Theorem 0.2.** Let $f_1, f_2, \ldots : X \to \mathbb{C}$ with each $f_k \in L^1(\mu)$ and assume that $\sum_{k=1}^{\infty} \int_X |f_k| \, d\mu < \infty$. Then $\sum_{k=1}^{\infty} f_k$ exists a.e. and

$$\int_X \left( \sum_{k=1}^{\infty} f_k \right) d\mu = \sum_{k=1}^{\infty} \int_X f_k \, d\mu.$$  

**Proof.** Let $g = \sum_{k=1}^{\infty} |f_k|$. Monotone convergence implies that $\int g = \int \sum_{k=1}^{\infty} |f_k| = \sum_{k=1}^{\infty} \int |f_k| < \infty$. Thus $g \in L^1(\mu)$ and so $g < \infty$ a.e. Thus, $\sum_{k=1}^{\infty} |f_k(x)| < \infty$ a.e. This implies that the series $\sum_{k=1}^{\infty} f_k(x)$. 

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converges absolutely a.e. Let \( F_j = \sum_{k=1}^{j} f_k \). Then \( F_j \) is dominated by \( g \) for all \( j \), and we can apply LDCT. We have

\[
\int_X \left( \sum_{k=1}^{\infty} f_k \right) d\mu = \int_X \lim_{j \to \infty} F_j \, d\mu = \lim_{j \to \infty} \int_X F_j \, d\mu = \lim_{j \to \infty} J_X \sum_{k=1}^{j} f_k \, d\mu = \sum_{k=1}^{\infty} \int_X f_k \, d\mu.
\]

\[ \square \]

**Countable additivity of the integral.** Let \( E_1, E_2, \ldots \) be a countable sequence of measurable sets. Let \( E = \bigcup_{k=1}^{\infty} E_k \) and \( f : X \to \mathbb{C} \) be measurable. Assume either \( f \geq 0 \) or \( f \in L^1(E) \) (i.e., \( \int_E f \, d\mu = \int_X f \chi_E \, d\mu < \infty \)). Then

\[
\int_E f \, d\mu = \sum_{k=1}^{\infty} \int_{E_k} f \, d\mu.
\]

**Proof.** First let \( f \geq 0 \). Then

\[
\int_E f \, d\mu = \int_X f \chi_E \, d\mu = \int_X \sum_{k=1}^{\infty} f \chi_{E_k} \, d\mu = \sum_{k=1}^{\infty} \int_X f \chi_{E_k} \, d\mu = \sum_{k=1}^{\infty} \int_{E_k} f \, d\mu.
\]

Now let \( f \in L^1(E) \) and \( f_k = \chi_{E_k} \). By the previous theorem, we need only check the convergence of the series of integrals of \( |f \chi_{E_k}| \).
We have
\[
\sum_{k=1}^{\infty} \int_{X} |f_k| \, d\mu = \sum_{k=1}^{\infty} \int_{X} |f| \chi_{E_k} \, d\mu
\]
\[
= \sum_{k=1}^{\infty} \int_{E_k} |f| \, d\mu
\]
\[
= \int_{E} |f| \, d\mu < \infty,
\]
because of the case when \( f \geq 0 \).