Proposition 0.1. Let $\mathcal{M}$ be a $\sigma$-algebra on $X$, let $Y$ be a topological space, and let $f : X \to Y$.

(a) Let $\Omega$ be a collection of sets $E \subset Y$ such that $f^{-1}(E) \in \mathcal{M}$. Then $\Omega$ is a $\sigma$-algebra on $Y$.
(b) If $f$ is measurable and $E \subset Y$ is Borel, then $f^{-1}(E) \in \mathcal{M}$.
(c) If $Y = [-\infty, \infty]$ (with open sets along with $[-\infty, a)$ and $(b, \infty]$ with $a, b \in \mathbb{R}$) and $f^{-1}((\alpha, \infty]) \in \mathcal{M}$ for all $\alpha \in [-\infty, \infty]$, then $f$ is measurable.

Proof. (a) Since $f^{-1}(Y) = X \in \mathcal{M}$, we have $Y \in \Omega$. Also $f^{-1}(E^c) = (f^{-1}(E))^c \in \mathcal{M} \Rightarrow E^c \in \mathcal{M}$. Lastly,
$$f^{-1}(\bigcup_{i=1}^{\infty} E_i) = \bigcup_{i=1}^{\infty} f^{-1}(E_i) \in \mathcal{M}.$$ (b) Because $f$ is measurable, all open sets are in $\Omega$. Since $\Omega$ is a $\sigma$-algebra, we have $\mathcal{B} \subset \Omega$.
(c) Recall $\Omega = \{E \mid f^{-1}(E) \in \mathcal{M}\}$. Given $\alpha \in \mathbb{R}$, choose $\alpha_n < \alpha$ so that $\alpha_n \to \alpha$ as $n \to \infty$. By assumption $(\alpha_n, \infty) \in \Omega$. Then
$$[-\infty, \alpha) = \bigcup_{n=1}^{\infty} [-\infty, \alpha_n] = \bigcup_{n=1}^{\infty} (\alpha_n, \infty)^c.$$ Thus $[-\infty, \alpha) \in \Omega$. Then $(\alpha, \beta] = [-\infty, \beta) \cap (\alpha, \infty] \in \Omega$. Hence, since $\Omega$ is a $\sigma$-algebra, $\Omega$ contains all open sets. It follows that $f$ is measurable.

Remark. All of these are equivalent:
$$f^{-1}([-\infty, \alpha)) \in \mathcal{M} \iff f^{-1}([-\infty, \alpha]) \in \mathcal{M} \iff f^{-1}([\alpha, \infty]) \in \mathcal{M} \iff f^{-1}((\alpha, \infty]) \& f^{-1}\{\{-\infty\}\} \in \mathcal{M}.$$ Limits. Let $\{a_n\}$ be a sequence in $\mathbb{R}$ or $[-\infty, \infty]$. Set
$$b_k = \sup\{a_k, a_{k+1}, \ldots\}, \ k = 1, 2, \ldots.$$ Then $\inf b_k = \lim \sup_{n \to \infty} a_n$. As $k$ gets larger, the sup is being taken over a smaller set, so $b_k \geq b_{k+1} \geq \ldots$. Thus $b_k$ is a (weakly) decreasing sequence and so $\lim b_k = \inf b_k$ exists. In other words,
lim sup is the largest limit point of the sequence (there exists a sub-sequence which converges to lim sup). Similarly, we could instead set $b_k = \inf\{a_k, a_{k+1}, \ldots\}$ and then $\sup b_k = \liminf_{n \to \infty} a_n$, that is, lim inf is the smallest limit point of the sequence. Note the relation $\liminf a_n = -\limsup\{-a_n\}$.

**Proposition 0.2.** A sequence $\{a_n\}$ converges if and only if

$$\liminf a_n = \limsup a_n = \lim a_n.$$  

**Limits of functions.** Let $\{f_n\}: X \to \mathbb{R}$ be a sequence of functions.

$$(\sup f_n)(x) = \sup\{f_n(x)\}$$

$$(\limsup f_n)(x) = \lim_{n \to \infty} f_n(x).$$

If, for each $x \in X$, the sequence $\{f_n(x)\}$ converges, then $f(x) = \lim f_n(x)$ is the pointwise limit. This works for $X = [-\infty, \infty]$ (convergence to $\pm \infty$ is obvious).

**Theorem 0.3.** If, for each $i = 1, 2, \ldots$, the function $f_i: X \to [-\infty, \infty]$ is measurable, then

$$g = \sup_{i>1} f_i \text{ and } h = \limsup_{n \to \infty} f_n$$

are both measurable.

**Proof.** NTS $g^{-1}((\alpha, \infty]) \in \mathcal{M}$ for all $\alpha$. We have

$$g^{-1}((\alpha, \infty]) = \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha, \infty]).$$

If $x$ is a member of the LHS, then $g(x) > \alpha$. Thus, some $f_n > \alpha$ from the definition of sup. If $x$ is a member of the RHS, then $f_i(x) > \alpha$ for some $i$, so $g \geq f_i(x) > \alpha$. Thus, $x \in g^{-1}((\alpha, \infty])$. Since $f_n^{-1}((\alpha, \infty]) \in \mathcal{M}$, and since countable unions are in $\mathcal{M}$, $g \in \mathcal{M}$. But then lim sup is measurable as well since by definition

$$\limsup f_k = \inf_{j \geq 1} \left(\sup_{k \geq j} f_k\right).$$

\[\square\]

**Corollary 0.4.** (a) Pointwise limits of measurable functions are measurable.

(b) If $f$ and $g$ are measurable, then $\max\{f, g\}$ and $\min\{f, mg\}$ are measurable.
Define “f plus” and “f minus” as follows:

\[ f^+ = \max\{f, 0\}, \quad f^- = -\min\{f, 0\} \]

Then \( f = f^+ - f^- \) and \( |f| = f^+ + f^- \).

**Simple functions.** A simple function is a function that takes only finitely many values in \( \mathbb{R} \) and does not take values \( \pm \infty \). Let \( \alpha_1, \ldots, \alpha_n \) be the values and \( A_i = \{x \in X \mid s(x) = \alpha_i\} \). Then

\[ s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}. \]

Note: \( \chi_{A_i} \) is measurable \( \iff \) \( A_i \in \mathcal{M} \). Constant function \( \alpha_i \) is measurable \( \Rightarrow \) product \( \alpha_i \chi_{A_i} \) is measurable. Since sums of measurable functions are measurable, \( s \) is measurable \( \iff \) all \( A_i \) are measurable.

**Theorem 0.5.** Let \( f: X \to [0, \infty) \) be measurable. Then there exists a sequence \( 0 \leq s_1 \leq s_2 \leq \ldots \) of measurable simple functions such that \( \lim_{n \to \infty} s_n = f \).

**Proof.** Partition \([0, n]\) into intervals of length \( 2^{-n} \). Define \( \varphi_n: [0, \infty] \to [0, \infty) \) as follows. Let \( \delta_n = 2^{-n} \). For each \( t \), choose \( k_n(t) \) such that \( k \delta_n \leq t < (k + 1)\delta_n \). Put

\[ \varphi_n(t) = \begin{cases} k_n(t)\delta_n & 0 \leq t < n; \\ n & n \leq t \leq \infty. \end{cases} \]

Note that each \( \varphi_n \) is Borel measurable, \( \varphi_1 \leq \varphi_2 \leq \cdots \leq t \) and \( \lim_{n \to \infty} \varphi_n(t) = t \). Let \( s_n = \varphi_n \circ f \). Then for any open set \( U \), \( s_n^{-1}(U) = f^{-1}(\varphi_n^{-1}(U)) = f^{-1}(\text{Borel set}) \in \mathcal{M} \). Thus \( s_n \) is measurable, increasing, and its limit is \( f \). \( \square \)

A positive measure is a mapping \( \mu: \mathcal{M} \to [0, \infty] \) which is countably additive:

\[ \mu \left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu(A_i) \text{ for disjoint } A_i. \]

**Lebesgue Integral.** Let \((X, \mathcal{M}, \mu)\) be a measure space. Let \( s: X \to [0, \infty) \) be the simple function \( s = \sum_{i=1}^{N} \alpha_i \chi_{A_i} \), where the \( A_i \) are disjoint. For each \( E \in \mathcal{M} \), define the integral

\[ \int_E s d\mu = \sum_{i=1}^{N} \alpha_i \mu(A_i \cap E). \]
For more general measurable functions $f : X \to [0, \infty]$, then
\[ \int_E f \, d\mu = \sup \left\{ \int_E s \, d\mu \mid s \text{ simple, } 0 \leq s \leq f \right\}. \]
If $f : X \to [-\infty, \infty]$, then
\[ \int_E f \, d\mu = \int_E f^+ \, d\mu - \int_E f^- \, d\mu, \]
provided both terms on the right-hand side are finite.