Riemann integral. If $s$ is simple and measurable then
\[ \int_X s \, d\mu = \sum_{i=1}^{N} \alpha_i \mu(E_i), \]
where $s = \sum_{i=1}^{N} \alpha_i \chi_{E_i}$. If $f \geq 0$, then
\[ \int_X f \, d\mu = \sup \left\{ \int_X s \, d\mu \mid 0 \leq s \leq f, s \text{ simple \& measurable} \right\}. \]

Recall the Riemann integral of function $f$ on interval $[a, b]$. Define lower and upper integrals $L(f, P)$ and $U(f, P)$, where $P$ is a partition of $[a, b]$. Set
\[ \int_X f = \sup_P L(f, P) \quad \text{and} \quad \int_X f = \inf_P U(f, P). \]

A function $f$ is Riemann integrable \iff \[ \int_X f = \int_X f, \]
in which case this common value is $\int f$.

A set $B \subset \mathbb{R}$ has measure zero if, for any $\epsilon > 0$, there exists a countable collection of intervals $\{I_i\}_{i=1}^{\infty}$ such that $B \subset \bigcup_{i=1}^{\infty} I_i$ and $\sum_{i=1}^{\infty} \lambda(I_i) < \epsilon$. Examples: finite sets, countable sets. There are also uncountable sets with measure zero. However, any interval does not have measure zero.

**Theorem 0.1.** A function $f$ is Riemann integrable if and only if $f$ is discontinuous on a set of measure zero.

A function is said to have a property (e.g., continuous) almost everywhere (abbreviated a.e.) if the set on which the property does not hold has measure zero. Thus, the statement of the theorem is that $f$ is Riemann integrable if and only if it is continuous almost everywhere.
Recall positive measure: a measure function \( \mu: \mathcal{M} \to [0, \infty] \) such that \( \mu \left( \bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} \mu(E_i) \) for \( E_i \in \mathcal{M} \) disjoint.

**Examples.**

1. “Counting measure.” Let \( X \) be any set and \( \mathcal{M} = \mathcal{P}(X) \) the set of all subsets. If \( E \subset X \) is finite, then \( \mu(E) = \# E \) (the number of elements in \( E \)). If \( E \subset X \) is infinite, then \( \mu(E) = \infty \).
2. “Unit mass at \( x_0 \) – Dirac delta function.” Again let \( X \) be any set and \( \mathcal{M} = \mathcal{P}(X) \). Choose \( x_0 \in X \). Set

\[
\mu(E) = \begin{cases} 
1 & \text{if } x_0 \in E; \\
0 & \text{if } x_0 \notin E.
\end{cases}
\]

**Theorem 0.2.**

1. If \( E \subset \mathbb{R} \) and \( \mu(E) < \infty \), then \( \mu(\emptyset) = 0 \).
2. (Monotonicity) \( A \subset B \Rightarrow \mu(A) \leq \mu(B) \).
3. If \( A_i \in \mathcal{M} \) for \( i = 1, 2, \ldots \), \( A_1 \subset A_2 \subset \cdots \), and \( A = \bigcup_{i=1}^{\infty} A_i \), then \( \mu(A_i) \to \mu(A) \) as \( i \to \infty \).
4. If \( A_i \in \mathcal{M} \) for \( i = 1, 2, \ldots \), \( A_1 \supset A_2 \supset \cdots \), \( \mu(A_1) < \infty \), and \( A = \bigcap_{i=1}^{\infty} A_i \), then \( \mu(A_i) \to \mu(A) \) as \( i \to \infty \).

**Proof.**

1. \( E = E \cup \emptyset \Rightarrow \mu(E) = \mu(E) + \mu(\emptyset) \).
2. \( B = A \cup (B \setminus A) \Rightarrow \mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A) \).
3. Let \( B_1 = A_1, B_2 = A_2 \setminus A_1, B_3 = A_3 \setminus A_2, \ldots \). Then the \( B_i \) are disjoint, \( A_n = B_1 \cup \cdots \cup B_n \), and \( A = \bigcup_{i=1}^{\infty} B_i \). Thus, \( \mu(A_n) = \mu(B_1) + \cdots + \mu(B_n) = \sum_{i=1}^{n} \mu(B_i) \), and (3) follows.
4. Let \( C_n = A_1 \setminus A_n \). Then \( C_1 \subset C_2 \subset \cdots \). We have \( \lambda(C_n) = \lambda(A_1) - \lambda(A_n) \). Also, \( A_1 \setminus A = \bigcup_n C_n \). Thus, \( A_1 \cap (\bigcap_i) = A_1 \setminus A_n \), and so

\[
\mu(A_1 \setminus A) = \lim_{i \to \infty} \mu(C_i) = \mu(A_1) - \lim_{i \to \infty} \mu(A_n).
\]

Hence, \( \mu(A_1) \to \mu(A) \).

**Properties of the Integral.**

(a) If \( 0 \leq f \leq g \) on \( E \), then \( \int_E f \, d\mu \leq \int_E g \, d\mu \).
(b) If \( A \subset B, A, B \in \mathcal{M} \), and \( f \geq 0 \), then \( \int_A f \, d\mu \leq \int_B f \, d\mu \).
(c) If \( f \geq 0 \) and \( c \in [0, \infty) \) is a non-negative constant, then \( \int_E c f \, d\mu = c \int_E f \, d\mu \).
(d) If \( f(x) = 0 \) for all \( x \in E \), then \( \int_E f \, d\mu = 0 \).
(e) If \( \mu(E) = 0 \), then \( \int_E f \, d\mu = 0 \).
(f) If \( f \geq 0 \), then \( \int_E f \, d\mu \leq \int_E \chi_E f \, d\mu \).

**Proof.**

(a) If \( s \leq f \) is simple, then \( s \leq g \) so the sup on \( g \) is over a larger class of simple functions than the sup on \( f \).
(b) We have $E_i \cap A \subset E_i \cap B$ for all $E_i$. If $s$ is simple,

$$\int_A s d\mu = \sum_{i=1}^N \alpha_i \mu(E_i \cap A) \leq \sum_{i=1}^N \alpha_i \mu(E_i \cap B) = \int_B s d\mu.$$

(c) For any simple $s$, $\int_E cs d\mu = c \int_E s d\mu$ since

$$\sum_i (c\alpha_i) \chi_{E_i} = c \sum i \alpha_i \chi_{E_i}.$$  

For any constant $c$, $s \leq f \iff cs \leq cf$. Thus,

$$\int cf = \sup_{s \leq cf} \int s = \sup_{s/c \leq f} \int s = \sup_{s' \leq f} \int cs' = c \int f.$$ 

(d) Let $s \leq f$ be simple and $s = \sum_{i=1}^N \alpha_i \chi_{E_i}$. Without loss of generality, $\alpha_1 = 0$ and $E_1 \supset E$. Thus,

$$\int_E s d\mu = \sum_{i=1}^N \alpha_i \mu(E_i \cap E) = \alpha_1 \mu(E) = 0.$$  

(The convention here and throughout is that $0 \cdot \infty = 0$.)

(e) If $s \leq f$ and $s = \sum \alpha_i \chi_{E_i}$, then $\int_E s = \sum_{i=1}^N \alpha_i \mu(E \cap E_i) = 0$.

(f) This could have been the definition of the integral. 

\[ \square \]