MEASURE AND INTEGRATION: LECTURE 9

Invariance of Lebesgue measure. Given $A \subset \mathbb{R}^n$ and $z \in \mathbb{R}^n$, let $z + A = \{z + x \mid x \in A\}$ be the translate of $A$ by $z$. Given $t > 0$, let $tA = \{tx \mid x \in A\}$ be the dilation of $A$ by $t$.

Let $I = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and $z = z_1 \times \cdots \times z_n$. Then

$$z + I = [z_1 + a_1, z_1 + b_1] \times \cdots \times [z_n + a_n, z_n + b_n],$$

and

$$tI = [ta_1, tb_1] \times \cdots \times [ta_n, tb_n],$$

and we have

$$\lambda(z + I) = (z_1 + b_1 - z_1 - a_1) \cdots (z_n + b_n - z_n - a_z)
= (b_1 - a_1) \cdots (b_n - a_n)
= \lambda(I).$$

and

$$\lambda(tI) = t^n \cdot \lambda(I).$$

If $P$ is a special polygon, then $\lambda(z + P) = \lambda(P)$ and $\lambda(tP) = t^n \lambda(P)$. Indeed, write $P = \sum_{i=1}^N I_i$ and the proof is straightforward.

If $G$ is an open set, then $\lambda(z + G) = \lambda(G)$ and $\lambda(tG) = t^n \lambda(G)$. We have $\lambda(G) = \sup\{\lambda(P) \mid P \subset G \text{ special polygon}\}$, so $\lambda(z + G) = \sup\{\lambda(P) \mid P \subset z + G, P \text{ special polygon}\}$. But $P \subset G$ special polygon $\iff z + P \subset z + G$ special polygon. Since Lebesgue invariance holds for special polygons, it holds for open sets.

Finally, by similar reasoning, it can be shown that a set $A \subset \mathbb{R}^n$ is measurable if and only if $z + A$ is measurable if and only if $tA$ is measurable, and $\lambda(A) = \lambda(z + A), \lambda(tA) = t^n \lambda(A)$.

A non-measurable set $E \subset \mathbb{R}^n$. Let $\mathbb{Q}$ be the set of rational numbers. For $x \in \mathbb{R}$, consider $x + \mathbb{Q} = \{x + q \mid q \in \mathbb{Q}\}$. Then $y \in x + \mathbb{Q} \iff y - x \in \mathbb{Q}$.

Claim: if $x, x' \in \mathbb{R}$, then either (i) $x + \mathbb{Q} = x' + \mathbb{Q}$ or (ii) $(x + \mathbb{Q}) \cap (x' + \mathbb{Q}) = \emptyset$. Proof: If the intersection is nonempty, then there exists $y = x + q_1 = x' + q_2$, which implies that $x - x' = q_1 - q_2 \in \mathbb{Q}$. Thus, $x + \mathbb{Q} = x' + \mathbb{Q}$, and the claim is proved.

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We have shown that $\mathbb{R}$ is covered disjointly by the sets $x + \mathbb{Q}$.

The Axiom of Choice states that there exists a set $E \subset \mathbb{R}$ such that every point of $\mathbb{R}$ belongs to only one of these sets, i.e.,

$$\mathbb{R} = \bigcup_{x \in E} (x + \mathbb{Q})$$

is a disjoint union. Alternatively, for any $x \in \mathbb{R}$, there exists a unique $y \in E$ and unique $z \in \mathbb{Q}$ such that $x = y + z$.

Since the set $\mathbb{Q}$ is countable, its elements can be enumerated: $\mathbb{Q} = \{q_1, q_2, \ldots\}$. Thus,

$$\mathbb{R} = \bigcup_{k=1}^{\infty} (q_k + E)$$

is a disjoint union. Using outer measure subadditivity and invariance of Lebesgue measure,

$$\lambda^*(\mathbb{R}) \leq \sum_{k=1}^{\infty} \lambda^*(q_k + E) = \sum_{k=1}^{\infty} \lambda^*(E).$$

Hence we must have that $\lambda^*(E) > 0$ (otherwise $\lambda^*(\mathbb{R}) = 0$).

Now let $K \subset E$ be an arbitrary compact subset of $E$ and let $D = (0, 1) \cap \mathbb{Q}$. (The set $D$ is a bounded countably infinite set.) Then

$$\bigcup_{q \in D} (q + K) = D + K$$

is a bounded set. The sets in the union are disjoint, since rational translates of $E$ are disjoint. We have

$$\infty > \lambda(D + K) \quad \text{(bounded)}$$

$$= \lambda\left(\bigcup_{q \in D} (q + K)\right)$$

$$= \sum_{q \in D} \lambda(q + K)$$

$$= \sum_{q \in D} \lambda(K).$$

Since the sum is over an infinite index set, $\lambda(K) = 0$. Because $K \subset E$ arbitrary $\Rightarrow \lambda(K) = 0$, we have $\lambda_*(E) = 0$. But $0 = \lambda_*(E) < \lambda^*(E) \Rightarrow E \notin \mathcal{L}$.

**Corollary 0.1.** If $A \subset \mathbb{R}^n$ is measurable with positive measure, then there exists $B \subset A$ that is not measurable.
Proof. Write \( A = \bigcup_{k=1}^\infty ((q_k + E) \cap A) \) as a disjoint union. Then
\[
0 < \lambda(A) = \lambda^*(A) \leq \sum_{k=1}^\infty \lambda^*((q_k + E) \cap A),
\]
and so \( \lambda^*((q_k + E) \cap A) > 0 \) for some \( k \). But \( \lambda^*((q_k + E) \cap A) \leq \lambda_*(q_k + E) = \lambda_*(E) = 0 \), a contradiction. \( \square \)

Invariance under linear transformations.

**Theorem 0.2.** Let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be a linear map and \( A \subset \mathbb{R}^n \). Then
\[
\lambda^*(TA) = |\det T| \lambda^*(A),
\]
\[
\lambda_*(TA) = |\det T| \lambda_*(A).
\]
If \( A \) is measurable, then \( TA \) is measurable and
\[
\lambda(TA) = |\det T| \lambda(A).
\]
Proof. First assume that \( T \) is invertible, i.e., that \( \det T \neq 0 \). We will use the following lemma.

**Lemma 0.3.** Let \( T \) be invertible and let \( J = [0,1)^n \). Let \( \rho \) be defined by \( \lambda(TJ) = \rho \lambda(J) \). If \( A \subset \mathbb{R}^n \), then \( \lambda^*(TA) = \rho \lambda^*(A) \) and \( \lambda_*(TA) = \rho \lambda_*(A) \). If \( A \) is measurable, then \( TA \) is measurable and \( \lambda(TA) = \rho \lambda(A) \).

Proof. The set \( J \) is the union of countably many compact sets:
\[
J = \bigcup_{k=1}^\infty [0, 1 - 1/k]^n,
\]
and so
\[
TJ = \bigcup_{k=1}^\infty T([0, 1 - 1/k]).
\]
Since \( T \) maps compact sets to compact sets, \( TJ \) is the union of countably many compact sets. Thus, \( TJ \) is measurable, so the definition of \( \rho \) makes sense.

We just need to prove that \( \lambda(TG) = \rho \lambda(G) \) for \( G \) open. As before, if the measure of open sets is invariant, then outer measure, compacts, and inner measures are invariant.

Let \( G \subset \mathbb{R}^n \) be open. Claim: can write \( G = \bigcup_{k=1}^\infty J_k \) with \( J_k \)’s disjoint and each \( J_k \) is a translation and dilation of \( J \). (Pair by integer of those not contained, then pair by \( 1/2 \), then by \( 1/4 \), ... ) Let \( J_k = z_k + t_k \cdot J \). Then \( \lambda(J_k) = t_k^n \lambda(J) \).
\[
TJ_k = Tz_k + t_k \cdot TJ
\]
\[ \Rightarrow \lambda(TJ_k) = t^\lambda_k \lambda(TJ) \]
\[ = t^\lambda_k \rho \lambda(J) \]
\[ = t^\lambda_k \rho t_k^{1-n} \lambda(J_k). \]
Thus, \[\lambda(TJ_k) = \rho \lambda(J_k).\] Since \( G = \bigcup_{k=1}^\infty J_k, \) \( TG = \bigcup_{k=1}^\infty TJ_k, \) which is a disjoint collection of measurable sets. Thus we have
\[ \lambda(TG) = \sum_{k=1}^\infty \lambda(TJ_k) = \sum_{k=1}^\infty \rho \cdot \lambda(J_k) = \rho \cdot \lambda(G). \]

\[ \square \]

To identify \( \rho, \) check for elementary matrices just on the cube. This shows that in fact \( \rho = |\text{det } T|. \)

Lastly, if \( T \) is not invertible, i.e., \( \text{det } T = 0, \) then the image \( T\mathbb{R}^n \) is the subset of a hyperplane. This means that \( TA \) has measure zero, so the formula still holds. \[ \square \]

A linear transformation is a rotation when the matrix is an orthogonal matrix: \( AAT = I. \) In this case, it must be that \( \text{det } A = \pm 1. \) Thus, Lebesgue measure is invariant under rotation.

Finally, there is an important subgroup of the group of all \( n \times n \) real matrices known as the special linear group, denoted
\[ SL(n, \mathbb{R}) = \{ A \mid \text{det } A = 1 \}. \]