Lecture 20: Holder continuity of Harmonic functions.

1 Holder continuity of Harmonic functions

In this lecture we will show that harmonic functions need to have a degree of regularity, specifically they must be Holder continuous.

Theorem 1.1 Let $L$ be a uniformly elliptic operator in divergence form taking

$$Lu = \frac{\partial}{\partial x_i} A_{ij} \frac{\partial u}{\partial x_j}. \quad (1)$$

If $u : \mathbb{R}^n \to \mathbb{R}$ is an $L$ harmonic function then $u$ is holder continuous.

The proof is a little involved, so we will first give a sketch of the proof, and then go back to fill in the details. The aim is to use Morrey’s lemma.

Proof Pick $x_0 \in \mathbb{R}^n$, and define the operator $\tilde{L}$ by

$$\tilde{L}f = \frac{\partial}{\partial x_i} A_{ij}(x_0) \frac{\partial f}{\partial x_j} = A_{ij}(x_0) \frac{\partial^2 f}{\partial x_i \partial x_j}. \quad (2)$$

Pick $s > 0$, and let $v$ be an $L$ harmonic function with $v = u$ on $\partial B_s(x_0)$. Note that the inequalities we proved in lecture 16 apply to $v$ so, in particular,

$$\int_{B_r(x_0)} |\nabla v|^2 \leq k \left( \frac{r}{s} \right)^n \int_{B_s(x_0)} |\nabla v|^2 \quad (3)$$

for all $r < s$. We use this and the inequality $(a + b)^2 \leq 2a^2 + 2b^2$ to estimate

$$\int_{B_r(x_0)} |\nabla u|^2 \leq 2 \int_{B_r(x_0)} |\nabla v|^2 + 2 \int_{B_r(x_0)} |\nabla (u - v)|^2 \quad (4)$$

$$\leq 2k \left( \frac{r}{s} \right)^n \int_{B_s(x_0)} |\nabla v|^2 + 2 \int_{B_r(x_0)} |\nabla (u - v)|^2 \quad (5)$$

$$\leq 2k \left( \frac{r}{s} \right)^n \int_{B_s(x_0)} |\nabla v|^2 + 2 \int_{B_s(x_0)} |\nabla (u - v)|^2. \quad (6)$$

Now use a lemma which we will prove later.
Lemma 1.2 Let $||A - A(x_0)|| = \sup_{B_r(x_0), i,j} |A_{ij} - A_{i,j}(x_0)|$. Then

$$\int_{B_r(x_0)} |\nabla (u - v)|^2 \leq \left( \frac{n||A - A(x_0)||}{\lambda} \right)^2 \int_{B_r(x_0)} |\nabla v|^2$$

(7)

and

$$\int_{B_r(x_0)} |\nabla (u - v)|^2 \leq \left( \frac{n||A - A(x_0)||}{\lambda} \right)^2 \int_{B_r(x_0)} |\nabla u|^2$$

(8)

By the first of these we get

$$\int_{B_r(x_0)} |\nabla u|^2 \leq \left( 2 \int_{B_r(x_0)} |\nabla v|^2 + 2 \int_{B_r(x_0)} |\nabla (u - v)|^2 \right) \int_{B_r(x_0)} |\nabla v|^2.$$  

(9)

Now we need to estimate this last integral in terms of $u$. We have

$$\int_{B_r(x_0)} |\nabla v|^2 \leq 2 \int_{B_r(x_0)} |\nabla u|^2 + 2 \int_{B_r(x_0)} |\nabla (u - v)|^2$$

(10)

$$\leq \left( 2 + 2 \left( \frac{n||A - A(x_0)||}{\lambda} \right)^2 \right) \int_{B_r(x_0)} |\nabla u|^2.$$  

(11)

by lemma 1.2. Plugging this back into ?? gives

$$\int_{B_r(x_0)} |\nabla u|^2 \leq \left( 2k' \left( \frac{r}{s} \right)^n + 2 \left( \frac{n||A - A(x_0)||}{\lambda} \right)^2 \right) \left( 2 + 2 \left( \frac{n||A - A(x_0)||}{\lambda} \right)^2 \right) \int_{B_r(x_0)} |\nabla u|^2.$$  

(12)

By choosing $s$ small we can get $n||A - A(x_0)||$ as small as we like. Therefore, for some constant $k'$ and for any $\delta > 0$ we can pick a small $s$ so that

$$\int_{B_r(x_0)} |\nabla u|^2 \leq \left( k' \left( \frac{r}{s} \right)^n + \delta \right) \int_{B_r(x_0)} |\nabla u|^2.$$  

(13)

We need one more lemma.

**Lemma 1.3** Let $\phi$ be a positive and increasing function on the positive reals, and let $\alpha, c$ be positive constants. For $0 < \gamma < \alpha$ there is $\delta > 0$ such that

$$\phi(r) \leq c_1 \left( \left( \frac{r}{s} \right)^\alpha + \delta \right) \phi(s)$$  

(14)

for $0 < r < s$ implies

$$\phi(r) \leq c_2 \left( \frac{r}{s} \right)^\gamma \phi(s),$$  

(15)

where $c_2$ is some constant that depends on $c_1, \alpha$ and $\gamma$.  

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In other words for any $0 < \gamma < \alpha$ we can prove by proving for a sufficiently small $\delta$. We will prove this later. Pick $0 < \beta < 1$ and apply this to with $\phi(r) = \int_{B_\gamma(x_0)} |\nabla u|^2$ and $\gamma = n - 2 + 2\beta$ to get

$$\int_{B_r(x_0)} |\nabla u|^2 \leq c \left( \frac{r}{s} \right)^{n - 2 + 2\beta} \int_{B_s(x_0)} |\nabla u|^2. \quad (16)$$

Let $C = \left( \frac{1}{s} \right)^{n - 2 + 2\beta} \int_{B_s(x_0)} |\nabla u|^2$, then

$$\int_{B_r(x_0)} |\nabla u|^2 \leq c \left( \frac{r}{s} \right)^{n - 2 + 2\beta} C, \quad (17)$$

so $u \in C^\beta$ by Morrey’s lemma. \hfill \Box

Now prove lemma’s 1.2 and 1.3.

**Lemma 1.2.** We wish to show that

$$\int_{B_s(x_0)} |\nabla (u - v)|^2 \leq \left( \frac{n\|A - A(x_0)\|}{\lambda} \right)^2 \int_{B_s(x_0)} |\nabla v|^2 \quad (18)$$

and

$$\int_{B_s(x_0)} |\nabla (u - v)|^2 \leq \left( \frac{n\|A - A(x_0)\|}{\lambda} \right)^2 \int_{B_s(x_0)} |\nabla u|^2. \quad (19)$$

**Proof** We will prove the first equation. The proof of the second is analogous. Calculate

$$\lambda \int_{B_s(x_0)} |\nabla (u - v)|^2 \leq \int_{B_s(x_0)} A_{ij} \frac{\partial (u - v)}{\partial x_i} \frac{\partial (u - v)}{\partial x_j} \quad (20)$$

$$\leq \int_{B_s(x_0)} A_{ij} \frac{\partial (u - v)}{\partial x_i} \frac{\partial u}{\partial x_j} - \int_{B_s(x_0)} A_{ij} \frac{\partial (v - u)}{\partial x_i} \frac{\partial v}{\partial x_j}. \quad (21)$$

Work on the first term. Clearly $\int_{\partial B_s(x_0)} (u - v)A \nabla u \cdot dS = 0$. By Stokes’ theorem we get

$$\int_{B_s(x_0)} A_{ij} \frac{\partial (u - v)}{\partial x_i} \frac{\partial u}{\partial x_j} = - \int_{B_s(x_0)} (u - v) \frac{\partial}{\partial x_i} A_{ij} \frac{\partial u}{\partial x_j} = \int_{B_s(x_0)} (u - v) Lu = 0. \quad (22)$$

Plugging this into gives

$$\lambda \int_{B_s(x_0)} |\nabla (u - v)|^2 \leq \int_{B_s(x_0)} A_{ij} \frac{\partial (v - u)}{\partial x_i} \frac{\partial v}{\partial x_j}. \quad (23)$$
By a similar calculation to ?? we get \( \int_{B_s(x_0)} A_{ij}(x_0) \frac{\partial (v-u)}{\partial x_i} \frac{\partial v}{\partial x_j} = 0 \), and
\[
\lambda \int_{B_s(x_0)} |\nabla (u-v)|^2 \leq \int_{B_s(x_0)} (A_{ij} - A_{ij}(x_0)) \frac{\partial (v-u)}{\partial x_i} \frac{\partial v}{\partial x_j} 
\leq ||A - A(x_0)|| \int_{B_s(x_0)} \sum_{i,j} \left| \frac{\partial (v-u)}{\partial x_i} \frac{\partial v}{\partial x_j} \right|.
\]
(24)

Now we need a minilemma, namely that if \( u, v \) are \( n \) vectors then \( \sum_{i,j} u_i v_j \leq n|u||v| \). Let \( w \) be the vector with \( w_i = v_1 + v_2 + \ldots + v_n \) for all \( i \). Note that
\[
|w| = \sqrt{n(v_1 + \ldots + v_n)^2} 
\leq n^{3/2} \sqrt{\frac{(v_1 + \ldots + v_n)^2}{n^2}} 
\leq n^{3/2} \sqrt{\frac{v_1^2 + \ldots + v_n^2}{n}} 
\leq n|v|
\]
(29)
since the square of the mean is less than or equal to the mean of the square. From this we get \( \sum_{i,j} u_i v_j = u \cdot w \leq |u||w| \leq n|u||v| \) as expected. Applying this to \( \nabla (u-v) \) and \( \nabla v \) gives
\[
\lambda \int_{B_s(x_0)} |\nabla (u-v)|^2 \leq n||A - A(x_0)|| \int_{B_s(x_0)} |\nabla (u-v)||\nabla v| 
\leq n||A - A(x_0)|| \left( \int_{B_s(x_0)} |\nabla (u-v)|^2 \right)^{1/2} \left( \int_{B_s(x_0)} |\nabla v|^2 \right)^{1/2}
\]
(30)
(31)
Finally divide and square to get
\[
\int_{B_s(x_0)} |\nabla (u-v)|^2 \leq \left( \frac{n||A - A(x_0)||}{\lambda} \right)^2 \int_{B_s(x_0)} |\nabla v|^2
\]
(32)
as required.

Lemma 1.3. We will show that if \( \phi \) is a positive and increasing function on \( \mathbb{R}^+ \) and
\[
\phi(r) \leq c_1 \left( \frac{r}{r'} \right) ^\alpha + \delta \phi(s)
\]
(33)
for \( r < r' \) and \( 0 < \delta < 1 \) then
\[
\phi(r) \leq c_2(\gamma) \left( \frac{r}{s} \right) ^\gamma \phi(s)
\]
(34)
where \( \gamma = \alpha \left( 1 + \frac{\log 2c_1}{\log s} \right) \), and \( c_2 \) is a constant depending on \( \gamma \).
Proof Choose $\tau = \delta^{1/\alpha}$ so that $\delta = \tau^\alpha$. Then

$$\phi(\tau s) \leq c(\tau^\alpha + \delta)\phi(s) \leq 2c\tau^\alpha\phi(s).$$  \hfill (35)

Therefore

$$\phi(\tau^k s) \leq (2c_1\tau^\alpha)^k \phi(s).$$  \hfill (36)

Pick $\gamma = \alpha \left(1 + \log \frac{2c_1}{\log \delta}\right)$ so that $2c_1\tau^\alpha - \gamma = 1$ and we have

$$\phi(\tau^k s) \leq \tau^{k\gamma}\phi(s).$$  \hfill (37)

When $r = \tau^k s$ this is precisely what we wanted with $c_2 = 1$. If instead $\tau^{k+1}s \leq r \leq \tau^k s$ then

$$\phi(r) \leq \phi(\tau^k s) \leq \tau^{k\gamma}\phi(s) \leq \frac{1}{r} \left(\frac{r}{s}\right)\gamma \phi(s)$$  \hfill (38)

which is what we needed. Finally note that by using a small $\delta$ we can get $\gamma$ as close as we like to $\alpha$ (though the constant will become nastier).