1 Adapting the proof of the gradient estimate to $\mathbb{R}^n$

Last time we proved a gradient estimate for solutions of the heat equation on a torus. In this lecture we will adapt the argument to prove the same theorem on $\mathbb{R}^n$.

**Theorem 1.1** If $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is positive and satisfies the heat equation then

$$t \left( \frac{|
abla u|^2}{u^2} - \frac{u_t}{u} \right) \leq \frac{n}{2}. \tag{1}$$

We will sketch the proof. We work on an interval $[0, T]$, and note that the result we want follows immediately from this. Define $f = \log u$ and $F = t \left( \frac{|
abla u|^2}{u^2} - \frac{u_t}{u} \right)$. Let $\phi$ be a cutoff function on the ball $B_1(0)$ with $0 < \phi < 1$ on the interior and $\phi = 0$ on the boundary. We can stretch this to get a cutoff function $\phi_r$ on $B_r(0)$ by taking $\phi_r(x) = \phi(x/r)$. If $F$ is non positive the result is trivially true, so we can assume that $\phi_r F$ has an interior maximum without loss of generality. At this maximum

$$\Delta(\phi_r F) \leq 0, \quad \frac{d(\phi_r F)}{dt} \geq 0, \quad \text{and} \quad \phi_r \nabla F = -F \nabla \phi_r \tag{2}$$

We’ll use these to get a bound on $F$. Calculate

$$0 \geq \left( \Delta - \frac{d}{dt} \right) (\phi_r F) \tag{3}$$

$$\geq \phi_r \Delta F + 2 \nabla F \cdot \nabla \phi_r + F \Delta \phi_r - \phi_r \frac{dF}{dt} \tag{4}$$

We need to estimate some of these. The calculations are very similar to last time. We start with
\[ \triangle F = t \Delta \left( \frac{\left| \nabla u \right|^2}{u^2} - \frac{u_t}{u} \right) \]  
(5)

\[ = t \Delta (|\nabla f|^2 - f_t) \]  
(6)

\[ = 2t \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)^2 + 2t \nabla (\triangle f) \cdot \nabla f - t \Delta f_t \]  
(7)

by Bochner. Calculate \( \triangle f = \frac{\partial}{\partial x_i} \frac{\partial u/\partial x_i}{u} = \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2} = \frac{F}{t} \) to get

\[ \triangle F = 2t \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)^2 - 2 \nabla F \cdot \nabla f - t \triangle f_t. \]  
(8)

Recall the inequality \( \sum A_{ii}^2 \leq n \sum (A_{ii})^2 \) for all matrices \( A \) from last time, and apply it to the hessian of \( f \) to give

\[ \triangle F = \frac{2t}{n} (\Delta f)^2 - 2 \nabla F \cdot \nabla f - t \triangle f_t. \]  
(9)

\[ \geq \frac{2F^2}{nt} - 2 \nabla F \cdot \nabla f - \triangle f_t. \]  
(10)

We also need an estimate on \( F_t \). We have

\[ F_t = |\nabla f|^2 - f_t + t(2 \nabla f \cdot \nabla f_t) - tf_{tt}, \]  
(11)

and \( \Delta f + |\nabla f|^2 = f_t \), so

\[ F_t = |\nabla f|^2 - f_t + t(2 \nabla f \cdot \nabla f_t) - t(\Delta f + |\nabla f|^2)_t \]  
(12)

\[ = \frac{F}{t} - t \Delta f_t. \]  
(13)

Putting 4, 10 and 13 together we get

\[ 0 \geq \phi_r \left( \frac{2F^2}{nt} - 2 \nabla F \cdot \nabla f - \frac{F}{t} \right) + 2 \nabla F \cdot \nabla \phi_r + F \Delta \phi_r. \]  
(14)

Recall that \( \phi_r \nabla F = -F \nabla \phi_r \), so

\[ 0 \geq \phi_r \left( \frac{2F^2}{nt} + \frac{2F}{\phi_r} \nabla \phi_r \cdot \nabla f - \frac{F}{t} \right) - \frac{2F}{\phi_r} |\nabla \phi_r| + F \Delta \phi_r \]  
(15)

\[ \geq F \phi_r \left( \frac{2F}{nt} + \frac{2}{\phi_r} \nabla \phi_r \cdot \nabla f - \frac{1}{t} - 2 \frac{|\nabla \phi_r|^2}{\phi_r^2} + \frac{\Delta \phi_r}{\phi_r} \right). \]  
(16)
Now use an absorbing inequality $\frac{\partial \phi_r}{\partial x_i}\frac{\partial f}{\partial x_i} \geq -\frac{1}{\epsilon} \left( \frac{\partial \phi_r}{\partial x_i} \right)^2 - \epsilon \left( \frac{\partial f}{\partial x_i} \right)^2$ for all $\epsilon > 0$. To show that

$$\nabla \phi_r \cdot \nabla f \geq -\frac{1}{\epsilon} |\nabla \phi_r|^2 - \epsilon |\nabla f|^2$$

(17)

for all $\epsilon > 0$. Consequently

$$0 \geq F \phi_r \left( \frac{2F}{nt} - \frac{2}{\phi_r} \left( \frac{1}{\epsilon} |\nabla \phi_r|^2 + \epsilon |\nabla f|^2 \right) - \frac{1}{t} - 2 \frac{|\nabla \phi_r|^2}{\phi_r^2} + \frac{\Delta \phi_r}{\phi_r} \right).$$

(18)

Let $r \to \infty$ so that $|\nabla \phi_r|$ and $\Delta \phi_r$ tend to zero and $\phi_r \to 1$, and get

$$0 \geq F \left( \frac{2F}{nt} - 2\epsilon |\nabla f|^2 - \frac{1}{t} \right).$$

(19)

Finally we let $\epsilon \to 0$ and recover

$$0 \geq \frac{F}{t} \left( \frac{2F}{n} - 1 \right).$$

(20)

From this we get $F \leq n/2$ as required.