1 The mean value inequality: Iterative argument continued

In this lecture we will complete the proof of the mean value inequality. Last time we had

\[
\int_{A_{r,k}} ((u-k)_+)^2 \leq \frac{\tilde{c}}{(R-r)^2(k-h)^{4/n}} \left( \int_{A_{R,h}} ((u-h)_+)^2 \right)^{(1+2/n)}
\]

(1)

for all \(h < k\) and \(r < R\). Define \(r_m = 1 + 2^{-m}\) and \(k_m = (2 - 2^{-m})k\) for some constant \(k\). Applying the above inequality to \(r_m, r_{m+1}, k_m, k_{m+1}\) gives

\[
\int_{A_{r_m+1,k_{m+1}}} ((u-k_{m+1})_+)^2 \leq \frac{\tilde{c}}{(r_m - r_{m+1})^2(k_{m+1} - k_m)^{4/n}} \left( \int_{A_{r_m,k_m}} ((u-k_m)_+)^2 \right)^{(1+2/n)}.
\]

(2)

Define \(\phi(m) = \left( \int_{A_{r_m,k_m}} ((u-k_m)_+)^2 \right)^{1/2}\) and \(\epsilon = 2/n\), so

\[
\phi(m+1) \leq \frac{\sqrt{c}}{(r_m - r_{m+1})(k_{m+1} - k_m)^{2/n}} (\phi(m))^{1+\epsilon}.
\]

(3)

Substituting for \(r_m, k_m\) and renaming \(c = \sqrt{c}\) gives

\[
\phi(m+1) \leq \frac{2^{m+1}c}{(2^{-m-1}k)^{2/n}} (\phi(m))^{1+\epsilon}.
\]

(4)

Now use induction to show that \(\phi(n) \to 0\) as \(n \to \infty\). Suppose there is some constant \(\gamma > 1\) with \(\phi(m) \leq \frac{\phi(0)}{\gamma^m}\). Then

\[
\phi(m+1) \leq \frac{2^{m+1}c}{(2^{-m-1}k)^{2/n}} \left( \frac{\phi(0)}{\gamma^m} \right)^{1+\epsilon}
\]

(5)

\[
\leq \left( \frac{2^{m+1}c}{(2^{-m-1}k)^{2/n}} \left( \frac{\phi(0)}{\gamma^m} \right) \right)^\epsilon \frac{\phi(0)}{\gamma^m}
\]

(6)
so if
\[ \left( \frac{2^{m+1}c}{(2-m-1k)^{2/m}} \left( \frac{\phi(0)}{\gamma^m} \right)^\epsilon \right) \leq \frac{1}{\gamma} \] (7)
then we get \( \phi(n) \leq \frac{\phi(0)}{\gamma^m} \) for all \( n \). It suffices to pick \( \gamma > 2^{1+\epsilon} \) and \( k > (2^{1+\epsilon}c\gamma(\phi(0))^\epsilon)^{\frac{1}{\epsilon}} = 2k'\phi(0) \) for appropriate \( k' \). Therefore

\[ \lim_{n \to \infty} \phi(n) \leq \lim_{n \to \infty} \frac{\phi(0)}{\gamma^m} = 0, \] (8)

so
\[ \int_{A_{r_m,k_m}} ((u - k_m)^+)^2 \to 0 \text{ as } n \to \infty. \] (9)

Note that \( \lim r_m = 1 \), and \( \lim k_m = 2k \) so we get
\[ \int_{A_{1,2k}} ((u - 2k)^+)^2 = 0 \] (10)

and conclude that \( u \leq 2k \) on \( B_1 \). Putting in our value for \( k \) we obtain
\[ \sup_{B_1(x_0)} u \leq (2^{1+\epsilon}c)^{\frac{1}{\epsilon}} \phi(0), \] (11)

and, writing out \( \phi(0) \) and \( \epsilon \),
\[ \sup_{B_1(x_0)} u \leq k' \left( \int_{A_{2,0}} (u^+)^2 \right)^{1/2} \] (12)
\[ \leq k' \left( \int_{B_2(x_0)} u^2 \right)^{1/2}. \] (13)

This is the mean value inequality.