1 Lecture Three: The Hopf Maximum Principle

In this lecture we will state and prove the Hopf Maximum Principle.

**Theorem 1.1** If \( u \) is an harmonic function on the closure of \( B_r(0) \subset \mathbb{R}^n \), and \( x_0 \) on the boundary of \( B_r(0) \) is a strict maximum of \( u \) (i.e. \( u(x_0) > u(y) \) for all \( y \neq x_0 \)) then

\[
\frac{\partial u}{\partial n}(x_0) \geq \frac{k}{r}(u(x_0) - u(0))
\]

for some strictly positive dimensional constant \( k \).

**Proof** We prove this from the maximum principle. First consider the case \( r = 1 \). Let \( v(x) = e^{-\alpha|x|^2} - e^{-\alpha} \), so \( v = 0 \) on \( \partial B_1(0) \) and \( v > 0 \) on the interior. Define

\[
w : \mathbb{R}^n \rightarrow \mathbb{R} \text{ by } w(x) = |x|^2
\]

and

\[
f : \mathbb{R} \rightarrow \mathbb{R} \text{ by } f(t) = e^{-\alpha t} - e^{-\alpha}
\]

so that \( v = f(w) \). Now consider

\[
\Delta f(w) = f''(w)|\nabla w|^2 + f'(w)\Delta w
\]

\[
= 4\alpha^2e^{-\alpha|x|^2}|x|^2 - 2n\alpha e^{-\alpha|x|^2}.
\]

Picking \( \alpha = 4n \) and restricting \( v \) to \( 1 \geq |x| \geq 1/2 \) we obtain

\[
\Delta v \geq 2\alpha e^{-\alpha}(\alpha/2 - n) \\
\geq 8n^2e^{-4n}.
\]

Now apply this to \( u \). On the annulus \( B_1 \setminus B_{1/2} \)
\[ \triangle(u + \epsilon v) = \epsilon \triangle v > 0, \] (2)

so \( u + \epsilon v \) is sub-harmonic on this annulus, and the maximum principle applies. Therefore the maximum of \( u + \epsilon v \) on the annulus \( B_1 \setminus B_{1/2} \) occurs on the boundary. Recall that \( u \) has a strict maximum on the outer boundary, so if we choose \( \epsilon \) very small we can arrange that \( u + \epsilon v \) also takes it’s maximum on the outer boundary. For this we need

\[ u(x_0) + \epsilon v(x_0) \geq \max_{\partial B_{1/2}} (u(x) + \epsilon v(x)) \]

so that

\[ u(x_0) \geq \max_{\partial B_{1/2}} u(x) + \epsilon (e^{-n} - e^{-4n}). \]

We can choose

\[ \epsilon = \frac{u(x_0) - \max_{\partial B_{1/2}} u(x)}{2(e^{-n} - e^{-4n})}. \] (3)

We know that \( u + \epsilon v \) has a maximum on the outer boundary and it has to be at \( x_0 \) (since \( v = 0 \) on the outer boundary). It follows that

\[ \frac{\partial (u + \epsilon v)}{\partial n} \geq 0 \]

and therefore that

\[ \frac{\partial u}{\partial n}(x_0) \geq -\epsilon \frac{\partial v}{\partial n}. \]

Calculating \( \frac{\partial u}{\partial n} \) and substituting in for \( \epsilon \) we obtain

\[ \frac{\partial u}{\partial n}(x_0) \geq -\frac{8n e^{-4n}}{2(e^{-n} - e^{-4n})} (u(x_0) - \max_{\partial B_{1/2}} u(x)). \] (4)

Finally we apply the Harnack inequality to get this in terms of \( u(0) \). Define \( w(x) \) by \( w(x) = u(x_0) - u(x) \). Note that \( w \) is harmonic and non-negative, therefore the Harnack inequality holds, and we get

\[ w(0) \leq \max_{B_{1/2}(0)} w(x) \leq C(n) \min_{B_{1/2}(0)} w(x) \]

for an appropriate dimensional constant \( C(n) \). Therefore

\[ \frac{u(x_0) - u(0)}{C(n)} \leq (u(x_0) - \max_{B_{1/2}(0)} u(x)). \] (5)

Substituting this into (4) we obtain

\[ \frac{\partial u}{\partial n}(x_0) \geq -\frac{8n e^{-4n}}{2C(n)(e^{-n} - e^{-4n})} (u(x_0) - u(0)). \] (6)
This completes the proof for \( r = 1 \). We will get the general case by scaling. If \( u \) is harmonic on \( B_r(0) \) and we define \( \tilde{u}(y) = u(xy) \) then \( \tilde{u} \) is harmonic on \( B_1(0) \). Also if \( x_0 \in \partial B_r(0) \) is a strict maximum of \( u \) then \( \tilde{x}_0 = x_0/r \) is a strict maximum of \( \tilde{u} \) on the boundary. Therefore

\[
\frac{\partial \tilde{u}}{\partial n}(\tilde{x}_0) \geq \frac{8ne^{-4n}}{2C(n)(e^{-n} - e^{-4n})}(\tilde{u}(\tilde{x}_0) - \tilde{u}(0)) \tag{7}
\]

\[
\geq \frac{8ne^{-4n}}{2C(n)(e^{-n} - e^{-4n})}(u(x_0) - u(0)). \tag{8}
\]

By the chain rule \( \frac{\partial \tilde{u}}{\partial n}(\tilde{x}_0) = r \frac{\partial u}{\partial n}(x_0) \), so

\[
\frac{\partial u}{\partial n}(x_0) \geq \frac{1}{r} \frac{8ne^{-4n}}{2C(n)(e^{-n} - e^{-4n})}(u(x_0) - u(0)) \tag{9}
\]

as required.